Automorphisms of the Torus

IDC451 Presentation

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Outline

Introduction

- The end goal of this talk is to introduce the Nielsen-Thurston Classification of the automorphisms of compact orientable surfaces.
- Rather than proving the theorem directly, which can be found in [CB88]or in [FM11], I would like to motivate it through the example of the Torus.
- The problem I would like to solve is the following:

Problem

Classifying the automorphism group of the Torus upto isotopy.

Outline

The Torus

Definition

Torus is defined as the topological space obtained by identifying the opposite sides of the unit square. This is usually denoted by T^2.

- There is an alternate (equivalent) way to define the Torus, which will be more useful to tackle our problem. Consider the space $\mathbb{R}^2$ and the additive group $\mathbb{Z}^2$. Define the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ as

$$(m, n) \cdot (x, y) = (x + m, y + n).$$

- Consider the equivalent relation

$$(x, y) \sim (x', y') \iff (x + m, y + n) = (x', y')$$

for some $(m, n) \in \mathbb{Z}^2$. The quotient under this equivalent relation, usually denoted as $\mathbb{R}^2/\mathbb{Z}^2$, is defined as T^2.

- Since any non-integer point (x, y) is equivalent to the point $(x - \lfloor x \rfloor, y - \lfloor y \rfloor)$, the points $(0, y) \sim (1, y)$, and $(x, 0) \sim (x, 1)$. Thus we get back our original definition of T^2 with the sides identified.

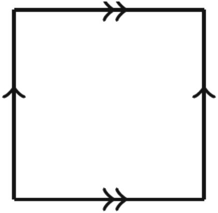

Figure: The Torus

Outline

Some Useful Concepts from Algebraic Topology

Homotopy and Isotopy

Let X, Y be a topological spaces and $f, g : X \to Y$ be continuous functions. A homotopy is a continuous function $H : [0,1] \times X \to Y$ such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$. When each $H(t,x)$, for a fixed t, is a homeomorphism then H is said to be an isotopy.

Example of Homotopy

Consider the maps $f, g : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, \sin(x))$ and $g(x) = (x^2, \cos(x))$ then

$$H(t,x) = ((1-t)x + tx^2, (1-t)\sin(x) + t\cos(x))$$

is a Homotopy from f to g.

Some Useful Concepts from Algebraic Topology

Path Homotopy

Let $\gamma, \sigma : [0,1] \to X$ be continuous maps, where X is a topological space. If γ, σ have the same base point and final point, i.e. $\gamma(0) = \sigma(0) = x_0$ and $\gamma(1) = \sigma(1) = x_1$, and if there exists a continuous map $H : [0,1]^2 \to X$ such that

$$H(0,x) = x_0,\ H(1,x) = x_1$$
$$H(t,0) = \gamma(t),\ H(t,1) = \sigma(t).$$

Then H is called a path Homotopy.

Consider curves $\gamma, \sigma : [0,1] \to X$ such that they start and end at x_0. Define *concatenation* of these curves as a new curve defined as

$$\sigma \circ \gamma(t) = \begin{cases} \gamma(2t),\ 0 \leq t \leq \frac{1}{2} \\ \sigma(2t-1),\ \frac{1}{2} \leq t \leq 1 \end{cases}$$

Some Useful Concepts from Algebraic Topology

Consider now the equivalence relation $\gamma \sim_p \sigma \iff \exists$ a path Homotopy between γ and σ. Clearly this is an equivalence relation.

Fundamental Group

Let $\pi_1(X, x_0)$ be the set of all closed curves at x_0 upto path Homotopy (i.e. quotiented out by $\sim_p$). Define a product on $\pi_1(X, x_0)$ by

$$[\sigma][\gamma] = [\sigma \circ \gamma].$$

It can be easily checked that this product is well defined and that $\pi_1(X, x_0)$ is a group under this product. This is called the Fundamental group.

Fundamental group of $\mathbb{R}$ is the trivial group, Fundamental group of S^1 is $\mathbb{Z}$, and the Fundamental group of $T^2 \simeq S^1 \times S^1$ is $\mathbb{Z}^2$ (product of topological spaces translates to product of π_1).

Induced homeomorphism of π_1

Let $f : X \to Y$ be a continuous map. Then f induces a group homeomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ given by $f_*([\gamma]) = [f \circ \gamma]$. Moreover if f is a homeomorphism then f_* is a group isomorphism.

The fundamental group of surfaces is generally finitely generated, so an automorphism of a surface can be studied by figuring out what the induced map does to the generators of the fundamental group.

Outline

Mapping Class Group

- Let S be a closed oriented surface and $\phi : S \to S$ be an orientation preserving diffeomorphism of S. The set of all such orientation preserving maps form a group under composition, denoted $\mathrm{Diff}_+(S)$.
- Consider the subgroup of $\mathrm{Diff}_+(S)$ given by the diffeomorphisms which are isotopic to the identity map on S, denote this by $\mathrm{Diff}_0(S)$.

Proposition

Let $\phi, \psi \in \mathrm{Diff}_+(S)$ then $\phi \simeq_{\text{iso}} \psi$ if and only if there exists a ξ such that $\phi = \xi \circ \psi$ and ξ is isotopic to the identity map.

Proof. ($\implies$) Suppose that $H(t, \cdot)$ is an isotopy from ϕ to ψ, the function $H(t, \psi^{-1})$ is an isotopy from $\xi = \phi \circ \psi^{-1}$ to the identity.
($\impliedby$) Conversely, if there exists some ξ which is isotopic to identity by the isotopy H', then $H(t, x) = H'(t, \psi(x))$ is an isotopy from ϕ to ψ.

Mapping Class Group

The above proposition shows that the quotient group

$$MCG(S) = \mathrm{Diff}_+(S)/\mathrm{Diff}_0(S)$$

is just the set of diffeomorphisms upto isotopy. This group is called the Mapping Class Group of the Surface S.

Theorem: Homeomorphisms and Diffeomorphisms

Let S be a closed orientable surface. Then every diffeomorphism of S is isotopic to some homeomorphism of S. [FM11]

This theorem tells us that classifying the homeomorphisms is the same as classifying the diffeomorphisms, upto isotopy. Thus determining $MCG(S)$ and classifying it is sufficient.

Outline

$MCG(T^2)$

- Consider a matrix $A \in GL_2(\mathbb{Z})$. Consider the map $\phi_A : T^2 \to T^2$ defined by

$$\phi_A((x,y) + \mathbb{Z}^2) = A(x,y)^T + \mathbb{Z}^2.$$

  It is easy to check that this map well defined.
- Since the inverse of A has to be divided by $\det(A) = ad - bc$, it follows that A is invertible in $GL_2(\mathbb{Z})$ if and only if $|\det(A)| = 1$. Since the orientation preserving maps have $\det(A) = +1$ we consider the group $SL_2(Z)$.
- The map ϕ_A is a diffeomorphism of T^2 if $A \in SL^2(\mathbb{Z})$.
- Suppose that $\phi : T^2 \to T^2$ is a homeomorphism in $MCG(T^2)$. Then the induced map $\phi_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is an isomorphism (recall that $\pi_1(T^2) = \mathbb{Z}^2$). Since this is invertible and ϕ is orientation preserving it follows that $\phi_* \in SL^2(\mathbb{Z})$.

$MCG(T^2)$

The following theorem summarizes this discussion clearly.

Theorem: $MCG(T^2)$

The map $\sigma : MCG(T^2) \to SL^2(\mathbb{Z})$ given by $\phi + \mathrm{Diff}_0(T^2) \mapsto \phi^*$ is an isomorphism of groups.

The proof of this is a bit involved, see [FM11].

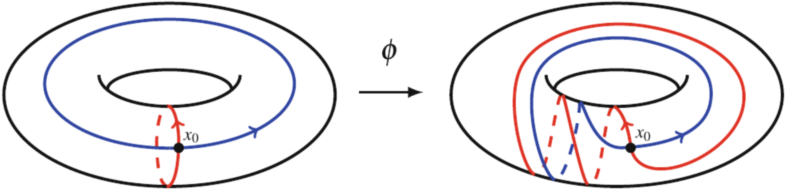

Figure: Homeomorphism's action on closed curves.

Classification of $MCG(T^2)$

- Consider an element $A \in SL_2(\mathbb{Z})$. Then,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ ad - bc = 1.$$

- The characterstic equation of such a matrix is $p(x) = x^2 - \tau x + 1$, where $\tau = a + d$ is the trace of the matrix. Thus it follows that the eigenvalues of A are given by $\lambda_\pm = (\tau \pm \sqrt{\tau^2 - 4})/2$.
- Thus there are three types of automorphisms of the torus, which correspond to the three cases of the eigenvalues:
  1. Complex eigenvalues, or $|\tau| < 2$,
  2. $\lambda_+ = \lambda_- = \pm 1$, or $|\tau| = 2$,
  3. Real distinct eigenvalues, or $|\tau| > 2$.

  We consider these case by case.

Classification of $MCG(T^2)$: Case 1

- We use Caley-Hamiltonian theorem repeatedly in each case. The theorem statement is the following

Caley-Hamilton Theorem

Let A be a matrix with characterstic equation p. Then $p(A) = 0$.

- Suppose that $\tau = 0$. By Caley-Hamiltonian theorem $A^2 + I_2 = 0$, or $A^4 = I_2$.
- In the case $\tau = -1$ the polynomial equation is $A^2 + A + I_2 = 0$, which upon multiplication with $(A - I)$ gives $A^3 = I_2$.
- In the case $\tau = 1$ we get the equation $A^2 - A + I = 0$. Multiplication by $(A^2 + A + I)(A^2 - I)$ we get $A^6 - I = 0$.
- Thus in general we can write that when $|\tau| < 2$ then the automorphism is such that $\phi^{12} \simeq_{\text{iso}} \text{id}$. The automorphism is said to be of finite order.

Classification of $MCG(T^2)$: Case 2

- In this case $|\tau| = 2$ and both the eigenvalues are ± 1. By Caley-Hamilton theorem $(A \pm I_2)^2 = 0$, meaning that $A = \pm I \pm N$, where N is some nilpotent matrix.
- The eigenvector of A is given by $g = (a \mp 1, c)$ when $c \neq 0$, and when $c = 0$ they are $g_\pm = (b, d \mp 1)$, depending on whether $\tau = \pm 2$.
- The elements $g_\pm \in \mathbb{Z}^2$ correspond to curves in $\pi_1(T^2)$ which are left invariant under the action of the automorphism corresponding to A.
- The two elements

$$T_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ T_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

  are particularly interesting. The eigenvectors of these are respectively $(0, 1)$ and $(1, 0)$, which are identified with the horizontal and vertical loops of the Torus.

Classification of $MCG(T^2)$: Case 2

- These elements generate $SL^2(\mathbb{Z})$ and are called *Dehn Twists* on $(0,1)$ and $(1,0)$ respectively. The action of T_1 is illustrated below.

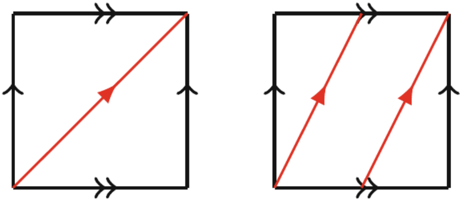

Figure: The action of Dehn twist T_1 on the curve $(1,1)$. The image is the curve $(1,2)$.

Classification of $MCG(T^2)$: Case 3

- In this case we have two distinct eigenvalues are $\lambda_{\pm} = (\tau \pm \sqrt{\tau^2 - 4})/2$, with $|\tau| > 2$.
- Thus the eigenvalues and eigenvectors are irrational in this case. This means that none of the curves are preserved by action of A.
- In this case the homeomorphism in $MCG(T^2)$ corresponding to A is a called an *Anosov* mapping.
- Most famous example of Anosov mapping is the Arnold's Cat map, which is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The action of this is given in the figure below. The larger eigenvalue in this case is ϕ^2 (the square of golden ratio).

Classification of $MCG(T^2)$: Case 3

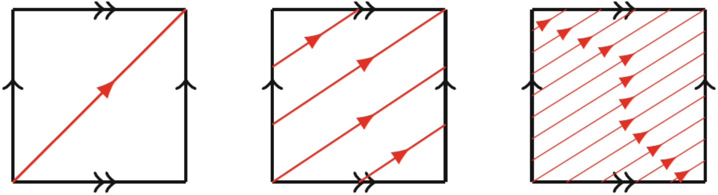

Figure: The action of Anosov map A on the curve $(1, 1)$. The image is the curve $(3, 2)$ in the middle figure, and $(8, 5)$ in the right figure. Since the slope is irrational the curve will end up being dense in T^2 without repeating.

Outline

Nielsen-Thurston Classification

In general the Nielsen-Thurston Classification theorem states the folowing.

Nielsen-Thurston Classification

Let S be a compact orientable surface. Given a homeomorphism $\phi : S \to S$, ϕ is isotopic to

1. A periodic automorphism,
2. or, an automorphism which preserves some finite union of simple closed curves,
3. or, a psuedo-Anosov mapping.

Here a psuedo-Anosov mappping is a generalization of the Anosov mapping to a general surface.

Outline

Conclusion

Figure: Homer Simpson.

References

A.J. Casson and S.A. Bleiler, *Automorphisms of surfaces after nielsen and thurston*, London Mathematical Society Student Texts, Cambridge University Press, 1988.

B. Farb and D. Margalit, *A primer on mapping class groups (pms-49)*, Princeton Mathematical Series, Princeton University Press, 2011.

Jean-Luc Thiffeault, *Braids and dynamics*, Series Title Frontiers in Applied Dynamical Systems: Reviews and Tutorials, Springer Nature, 2022.