On the Geodesics of Thurston's Asymmetric Metric

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Teichmüller Space

A pair (X, f) where X is a hyperbolic surface and $f : S \to X$ is a homeomorphism is called a *marking* of S. Two markings (X, f) and (Y, g) are said to be equivalent if there is an isometry $I : X \to Y$ such that $If \simeq_{\text{iso}} g$. The *Teichmüller space* of S is defined as the space of a hyperbolic markings on S up to the above equivalence.

$$\mathrm{Teich}(S) = \{(X, f : S \to X) \mid f \text{ is orientation preserving}\} / \sim$$

We denote the elements of $\mathrm{Teich}(S)$ by $\mathfrak{X}$.

For example for the pair of pants, i.e. sphere with three punctures $S_{0,3}$ the Teichmüller space is homeomorphic to $\mathbb{R}^3_+$.

Thurston Metric

In his paper, Thurston asks the following question: given a surface S with two hyperbolic structures $f : S \to X$ and $g : S \to Y$, is there a homeomorphism $\varphi : X \to Y$ compatible with the markings which realizes the least possible value of the Lipschitz constant? In other words if

$$L = \inf_{\substack{\psi : X \to Y \\ \psi f \simeq g}} \mathrm{Lip}(\psi) \tag{1}$$

then does there exist a $L-$Lipschitz homeomorphism φ. It turns out that the answer to this question is positive. The definition of the Thurston metric was motivated by this question. Let $L : \mathrm{Teich}(S) \times \mathrm{Teich}(S) \to \mathbb{R}_+$ be defined as

$$L(\mathfrak{X}, \mathfrak{Y}) = \inf_{\substack{\psi : X \to Y \\ \psi f \simeq g}} \log(\mathrm{Lip}(\psi)) \tag{2}$$

This is called Thurston's asymmetric metric. Thurston had also defined another metric on $\mathrm{Teich}(S)$ as follows: define $K : \mathrm{Teich}(S) \times \mathrm{Teich}(S) \to \mathbb{R}_+$ as

$$K(\mathfrak{X}, \mathfrak{Y}) = \sup_{c \in \mathcal{S}} \log\left(\frac{\ell_{\mathfrak{Y}}(c)}{\ell_{\mathfrak{X}}(c)}\right) \tag{3}$$

Thurston had showed that these two metrics are equal!

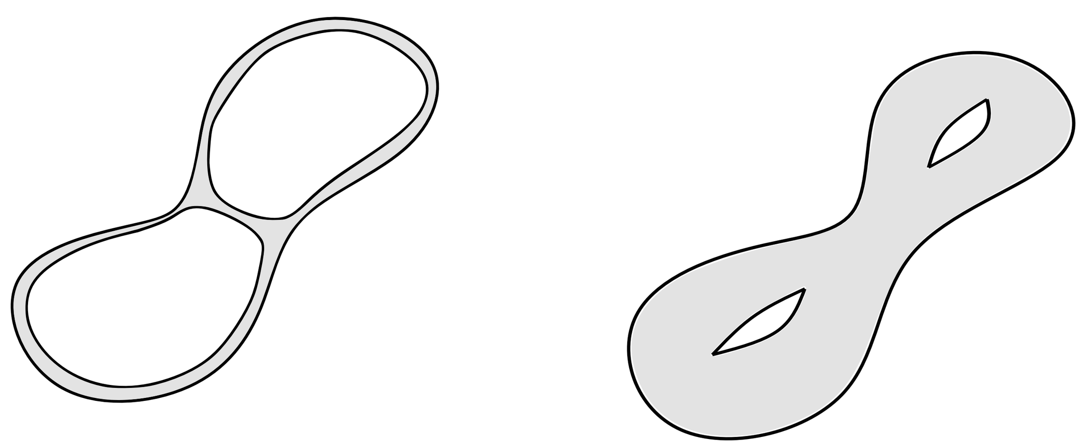


Fig. 1: Points far away in one direction, but close in the other.

Geodesics

What do geodesics in Thurston's metric look like?

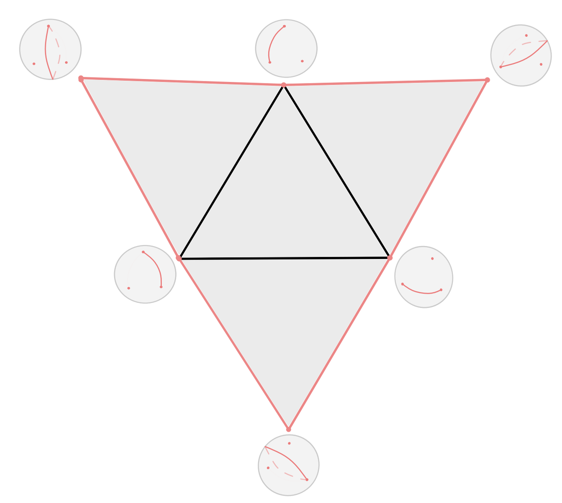
Thurston had given an example of geodesics called *Stretch Rays*. This involved "cutting up" the surface into ideal hyperbolic triangles, constructing a pair of foliations on these polygons, and explicitly constructing Lipschitz maps which stretches along one of these foliations.

Arc Complex

Let Σ be a topological surface with boundary. The arc complex $\mathscr{A}(\Sigma, \partial\Sigma)$ of Σ is a simplicial complex defined as follows:

1. The $0-$simplexes are homotopy classes of simple essential arcs relative to the boundary.
2. The vertices $(\alpha_1, \cdots, \alpha_n)$ span an $n-$simplex if $\underline{\alpha} = \bigcup_{i=1}^{n} \alpha_i$ is an *arc system*.

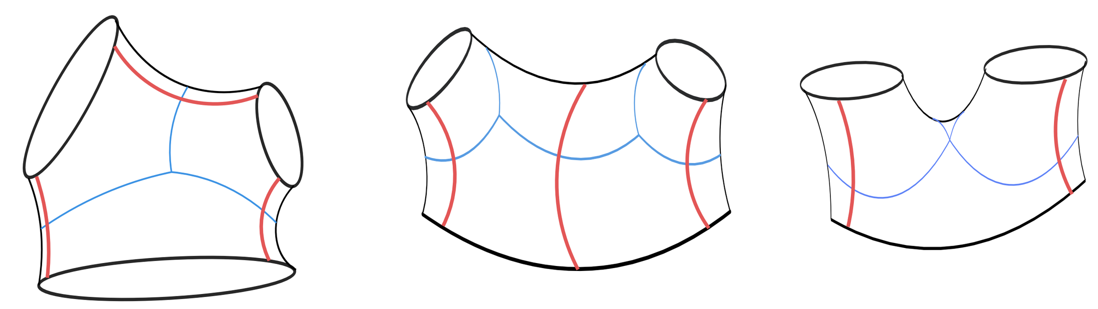
The sub-complex $\mathscr{A}_\infty(\Sigma, \partial\Sigma)$ of $\mathscr{A}(\Sigma, \partial\Sigma)$ only has simplexes whose vertices form a non-filling arc systems. The compliment of the non-filling arc complex is called the filling arc space and denoted $\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)$. The geometric realization space of $\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)$, denoted $|\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)| \times \mathbb{R}^+$, is the space of all weighted filling arc systems.



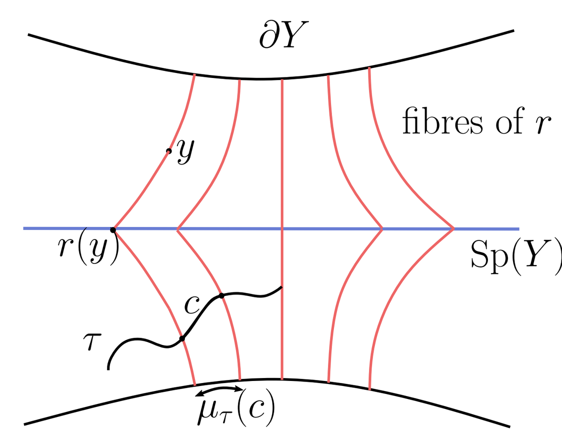
Dual Arcs of a Surface with Boundary

Let Y be a hyperbolic surface with geodesic boundary which is homeomorphic to Σ. The *valency* of a point $y \in Y$ is the cardinality of the set $\{p \in \partial Y \mid d(y, p) = d(y, \partial Y)\}$. The *spine* of a hyperbolic surface is defined as $\mathrm{Sp}(Y) = \{y \in Y \mid \text{valency of } y \geq 2\}$. This is called the spine as it is a deformation retract of Y. There is a natural deformation retract $r : Y \to \mathrm{Sp}(Y)$ s.t. the fibers are geodesic arcs orthogonal to the boundary.

Given any two points $y, y' \in \mathrm{Sp}_2(Y)$ which lie on the same edge e of the spine the arcs $r^{-1}(y)$ and $r^{-1}(y')$ are homotopic to each other relative to the boundary ∂Y. A dual arc α_e is the homotopy class relative to ∂Y of fibers of r corresponding to e. There is a special representative of the class α_e which is the fiber perpendicular to e as well as the boundary.



If τ is an arc in Y transverse to the fibers of r such that $r(\tau) \subset e$ for some edge e of $\mathrm{Sp}(Y)$ then define a measure μ_τ on τ by defining the measure of any sub-arc c to be length of the curve on ∂Y obtained by continuously deforming c to the boundary keeping each point of c on the same fiber of r. For an arbitrary transverse arc τ, it can be decomposed into finitely many components.



Weighted Filling Arc Space is Teichmuller space

Every surface can be assigned a weighted arc system by defining $\underline{A}(Y) = \sum_{e \subset \mathrm{Sp}(Y)} \mu_e(e)\alpha_e$. Note that if Y and Y' represent the same point in $\mathrm{Teich}(\Sigma)$ then the isometry preserves the weighted dual arc system. This means that $\underline{A}(\square)$ can be defined as a function on $\mathrm{Teich}(S)$. The following theorem was proved by Luo and later, in more generality, by Calderon and Farre.

The map $\underline{A} : \mathrm{Teich}(S) \to |\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)| \times \mathbb{R}^+$ is a homeomorphism.

Dilation Rays

Let $\mathfrak{X} \in \mathrm{Teich}(S)$ and λ be a multicurve in X. Consider the completion $Y_1, \cdots, Y_n$ of the components of $X - \lambda$ and the corresponding points $\mathfrak{Y}_i \in \mathrm{Teich}(\Sigma_i)$. Then $\underline{A}^{-1}(e^t \underline{A}(\mathfrak{Y}_i))$ defines a curve in $\mathrm{Teich}(\Sigma_i)$ denoted by $\mathfrak{Y}_i(t)$. Then gluing all $Y_i(t)$ together without any twisting gives a curve $\mathfrak{X}_t^\lambda$ in $\mathrm{Teich}(S)$. This curve is called the *Dilation ray* based at X.

We have proved the following result:

If $\mathfrak{X} \in \mathrm{Teich}(S)$ and λ is a pants decomposition of S, then the dilation ray $\mathfrak{X}_t^\lambda$ is a Thurston geodesic.