ON RETRACTS OF FREE GROUPS

Presentation for Combinatorial Group Theory Course

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1 Retraction: Equivalent definitions

Definition 1.1. A retract H of a group G is a subgroup such that there is a homomorphism $r : G \to H$ such that $r(h) = h$ for all $h \in H$.

Proposition 1.2. *H is a retract of G if and only if every homomorphism $f : H \to K$, for any group K, can be extended to a homomorphism G to K.*

Proof. Suppose that H is a retract. Then there is some $r : G \to H$ such that $r(h) = h$ for all h in H. The composition $\tilde{f} := f \circ r : G \to K$ such that $\tilde{f}(h) = f \circ r(h) = f(h)$. Conversely, consider the identity morphism on H. It can be extended to a homomorphism $\tilde{f} : G \to H$, which is a retract. □

Proposition 1.3. *H is a retract of a G if and only if there exists a normal subgroup of G such that $G = HN$ and $H \cap N = 1$.*

Proof. Suppose that H is a retract of G. Then consider the kernel, N, of the retraction map $r : G \to H$. If $h \in H \cap N$ then $h = r(h) = 1$. Thus $H \cap N = 1$. Let $g \in G$ and let $h = r(g)$. Then $h^{-1}g \in N$ since $r(h^{-1}g) = h^{-1}r(g) = 1$. Therefore $G = HN$. Conversly, let $r : G \to H$ be the map $r(g) = h$ where $g = hn$. This map is well defined since if $h_1n_1 = h_2n_2$ then $h_2^{-1}h_1 = n_2n_1^{-1}$, but since $H \cap N = 1$ it follows that $h_1 = h_2$ and $n_1 = n_2$. It is a homomorphism because $r(h_1n_1h_2n_2) = r(h_1h_2(h_2^{-1}n_1h_2)n_2) = h_1h_2$ (since the bracketted term is in N as well). Since by definition $r(h) = h$ for all h in H it follows that r is a retract. □

Some examples of retracts. Consider any free group $F(x_1, \cdots, x_n)$. Consider the subgroup H which is freely generated by $x_1, \cdots, x_r$. Then $r : F \to H$ which sends $x_i \mapsto x_i$ when $i \leq r$ and $x_i \mapsto 1$ when $i > r$, is a retract. In the next section we will classify all the retracts of a free group of finite order.

Consider the free group $F(a, b)$ and the subgroup $H = \langle ab^2 \rangle$. The map $r : F(a, b) \to H$ given by $a \mapsto ab^2$ and $b \mapsto 1$. This is retract of the free group, and note that H is not generated by any subset of the generators.

The general question we want to answer is what are all the retracts of Free groups? As seen by the above example they are not just the ones which are generted by a subset of the generators of the free group.

2 Nielson Transformation

Definition 2.1. Let F be a free group and $X = \{x_1, \cdots, x_n\}$ be a freely generating set. A Nielsen transformation is a transformation of the form

1. The cyclic permutation $\{x_1, \cdots, x_n\} \to \{x_n, x_1, \cdots, x_{n-1}\}$.
2. Interchanging of x_1 and x_2.
3. Replace x_1 with x_1^{-1}.
4. Replace x_1 with $x_1 x_2$.
5. Delete x_1 if it is 1.

The following is the key theorem proven by Nielsen, but I will not prove it here.

Theorem 2.2. The Nielsen transformations freely generates $Aut(F_n)$. Thus F_n is finitely presented.

Theorem 2.3. If F is a finite rank free group generated by $x_1, \cdots, x_n$ and F/N is also a free group, then there exists a freely generating set $a_1, \cdots a_n$ such that $a_{r+1}, \cdots, a_n$ generates N.

Proof. Let $\bar{b}_1, \cdots, \bar{b}_r$ freely generates F/N, and let π be the natural projection from F to F/N. Let $\bar{x}_i = \pi(x_i) = w(\bar{b}_\nu)$. Since $\bar{x}_i$ generates F/N it follows that there is a regular Nielsen transformation N_α given by

$$N_\alpha(w_1(\bar{b}_\nu), \cdots, w_n(\bar{b}_\nu)) = (\bar{b}_1, \cdots, \bar{b}_r, 1, \cdots, 1)$$

Applying the same set of Nielsen transformations to $x_1, \cdots, x_n$ in F

$$N_\alpha(x_1, \cdots, x_n) = (a_1(x_\mu), \cdots, a_n(x_\mu)).$$

Since N_α is a regular Nielsen transformation it follows that a_i generate F. Also,

$$\pi(a_i(x_\mu)) = a_i(\pi(x_\mu)) = a_i(w_\mu(\bar{b}_\nu)) = \begin{cases} \bar{b}_i, \ i \leq r \\ 1, \ i > r \end{cases}$$

This shows that $a_i \in N$ for $i > r$. We need show that a_i freely generates N for $i > r$. Let N' be the normal subgroup generated by $a_{r+1}, \cdots, a_n$, and let $u(a_\nu)$ be a word in N. Then the quotient F/N' is generated by $a_1, \cdots, a_r$. This means that

$$u(a_\nu) = s(a_1, \cdots, a_r)t(a_\nu)$$

where s is a reduced word and t is a word in N'. Thus,

$$1 = \pi(u(a_\nu)) = \pi(s(a_1, \cdots, a_r)) = s(\bar{b}_1, \cdots, \bar{b}_r).$$

Since $\bar{b}_i$ are free generators and the word was reduced, it follows that s is the empty word and therefore $u(a_\nu) = t(a_\nu) \in N'$. Hence $N \leq N'$, but N contains $a_{r+1}, \cdots, a_n$ thus $N = N'$. $\square$

3 Characterization of the retracts of finitely generated free groups.

Theorem 3.1. Let F be a free group of rank $n \leq \infty$. Then H is a retract of F_n if and only if there is a generating set $F = F(a_1, \cdots, a_n)$ such that $H = \langle h_1, \cdots, h_r \rangle$ where $h_i = a_i n_i$, where n_i is in the normal subgroup generated N by $a_{r+1}, \cdots, a_n$.

Proof. It is clear that if $H = \langle h_1, \cdots, h_r \rangle$ where $h_i = a_i n_i$ with n_i in N. Then the map $r : F \to H$ which maps $a_i \mapsto a_i n_i$ when $i \leq r$ and $a_i \mapsto 1$ when $i > r$. This is clearly a retraction since $r(n_i) = 1$ becasue it is in the normal subgroup generated by a_i for $i > r$.

Conversely, suppose that H is a retract of F, and suppose that rank of H is r. Since H is a retract there exists a normal subgroup N such that $F = HN$ and $H \cap N = 1$. F/N is free since $F/N = HN/N \cong H/H \cap N = H$. By the theorem above it follows that there is a generating set $a_1, \cdots, a_n$ of H such that $a_{r+1}, \cdots, a_n$ generates N. The fact that $F = HN$ can be used to write a_i as $h_i n_i^{-1}$ for some $n_i \in N$ where $i \leq r$. Thus we have $h_i = a_i n_i$ where $i \leq r$. h_i freely generate H since they are the images of $a_i N$ (under the natural isomorphism) which freely generates F/N. $\square$

Remark. Note that this classification does not work for infinite rank free groups. Mainly because Nielson transformations do not generate the automorphism group of infinite rank free groups. The proof heavily relies on that theorem. An interesting counter example would be coming up with a retract of F_∞ which is not of the form as given in the above theorem.

References

[1] D. S. Wilhelm Magnus, Abraham Karrass, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, 2nd ed., ser. Dover Books on Mathematics. Dover Publications, 2004.