Dilation Rays in Teichmuller Space

Master's Thesis Project

Manvendra Somvanshi

[1]Indian Institute of Science Education Research, Mohali

[2]Max Planck Institute for Mathematics in Sciences

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- Most well known examples of geodesics in Thurston's metric are the Stretch maps in [Thu98].
- The goal of my master's thesis is to prove that Dilation rays, which generalizes the Stretch maps to arbitrary laminations, are geodesics as well.
- I will introduce the concepts of orthogeodesic foliation, dual arcs, and, filling arc complex in order to define the Dilation rays.

Teichmuller Space of Hyperbolic Structures

Let S be a closed surface of genus $g \geq 2$ then the Teichmuller space of S is

$$\begin{aligned}\mathcal{T}(S) = \{(X, f : S \to X) \mid & X \text{ is a hyperbolic surface and} \\ & f \text{ is an o.p. homeomorphism}\}/\sim \qquad (1)\end{aligned}$$

where the quotient is by the equivalence relation $(X, f) \sim (Y, g)$ if there is an isometry $I : X \to Y$ such that $I \circ f \simeq g$ (isotopic).

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "I"'] & & Y \end{tikzcd}$$

On $\mathcal{T}(S)$, Thurston's metric is defined as

$$d_{Th}([(X,f)],[(Y,g)]) = \inf_{\substack{\phi:X\to Y,\\ \phi\circ f\simeq g}} \log(Lip(\phi)) \tag{2}$$

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This is a metric as shown in [Thu98]. More over Thurston showed in [Thu98] that this metric is the same as:

$$d_{Th}([(X,f)],[(Y,g)]) = \sup_{\alpha \in \text{s.c.c.}} \log\left(\frac{\ell_Y(g(\alpha))}{\ell_X(f(\alpha))}\right) \tag{3}$$

Spine of a Surface

Let Y be a compact hyperbolic surface with geodesic boundary and Σ be the underlying topological space.

- The valency of a point $y \in Y$ is $val(y) = \#\{p \in \partial Y \mid d(y, p) = d(y, \partial Y)\}$.

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- The valency of a point $y \in Y$ is $val(y) = \#\{p \in \partial Y \mid d(y,p) = d(y, \partial Y)\}$.
- Define the spine of a surface $Sp(Y) = \{y \in Y \mid val(y) \geq 2\}$. Let $Sp^k(Y) \subset Sp(Y)$ be the subset which only contains the points with valency exactly k.

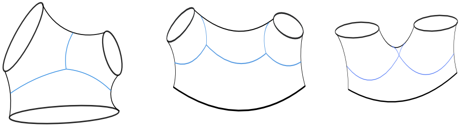

Figure: Spine of a Pair of Pants in three different metric structures.

Orthogeodesic Foliation

From each point $y \in Sp^k(Y)$ there are $k-$geodesics from the point which are perpendicular to the boundary. These arcs give a foliation with $k-$pronged singularities where $k \geq 3$ on the surface Y, called the orthogeodesic foliation. We denote this by $\mathcal{O}_{\partial Y}(Y)$.

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- Let γ be a curve that intersects the foliation transeversely such that it can be transversely isotoped into a segment of the boundary with end points staying on the same leaves of the foliation. We define $\mu(\gamma)$ as the length of γ at the boundary.

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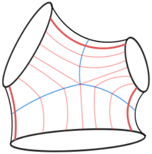
- For an arbitrary transveres arc, cut it such that each component is transversely isotopic to the boundary.



Dual Arcs

Let $r : Y \to Sp(Y)$ be the natural deformation retract collapsing Y to the spine along the orthogeodesic leaves.

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Given any two points $x, y \in e \subset Sp(Y)$ where e is an edge of $Sp(Y)$, then the leaves $r^{-1}(x)$ and $r^{-1}(y)$ are isotopic with end points staying on the same boundary component of Y.

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We represent by α_e both the unique leaf of $\mathcal{O}_{\partial Y}(Y)$ which is perpendicular to the edge e and the homotopy class of $r^{-1}(x)$ with the end-points staying on the same boundary component. We call α_e the dual arc to e.

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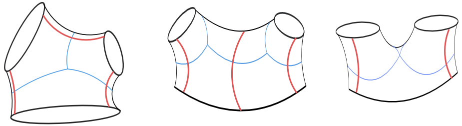

Figure: Dual arc systems of a Pair of Pants in three different metric structures.

Filling Arc Systems

A filling arc system on a surface is a disjoint collection of (isotopy classes of) arcs whose compliment is a union of contractible surfaces. Note that when the surface itself is contractible, then the empty arc system is a filling arc system.

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The dual arc system $\underline{\alpha}(Y) = \bigcup_{e \in Sp(Y)} \alpha_e$ is filling.

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proof

This just follows from the fact that each component of $Y - \underline{\alpha}(Y)$ deformation retracts to a contractible subset of $Sp(Y)$.

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- These vertices $\underline{A}_1, \cdots, \underline{A}_n$ span a simplex if there exists representatives of the isotopy classes of the arcs in $\underline{A}_k$ for all k, such that no two arcs intersect.
- Let $|\mathscr{A}(\Sigma, \partial\Sigma)|_{\mathbb{R}}$ be the geometric realization of the complex in Euclidean space, and $|\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)|_{\mathbb{R}}$ be the subspace where each point corresponds to a filling arc system.

Dilation Rays

Given a hyperbolic structure $[(Y, f)]$ on Σ we have a weighted arc system $\underline{A}(Y) = \sum_{e \in Sp(Y)} \mu(e)\alpha_e \in |\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)|_{\mathbb{R}}$.

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Theorem [CF24]

The map $\underline{A} : \mathcal{T}(\Sigma) \to |\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)|_{\mathbb{R}}$ which associates to each element Y the weighted dual arc system is a homeomorphism.

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Dilation Rays

Let $X \in \mathcal{T}(S)$ with a lamination λ, $Y \in \mathcal{T}(\Sigma)$ be the metric completion of some component of $X - \lambda$, and $\underline{A}(Y) = \sum c_e \alpha_e$. Let Y_t based at Y is the pullback of the ray $\sum e^t c_e \alpha_e \in |\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)|_{\mathbb{R}}$ by $\underline{A}$. For each t we can glue up the components without twisting, to get the closed surface X_t, and a foliation $\mathcal{O}_\lambda(X)$. The ray X_t is called Dilation ray.

Original Results

Prof. Farre at MPI-MIS and I have proved the following two results.

Theorem A

Let Σ be a pair of pants. If $[(Y, f)] \in \mathcal{T}(\Sigma)$ and $\underline{A}(Y) = \sum_{i=1}^{n} c_i \alpha_i$. If $Y_t = \underline{A}^{-1}(e^t \underline{A}(Y))$ then there is a e^t-Lipschitz map $f_t : Y \to Y_t$ such that the dual arcs of Y map to the dual arcs of Y_t and is identity on the boundary.

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Theorem B

If X is a hyperbolic surface and λ is a geodesic lamination which is a limit of a sequence of pants decomposition of X. Then the dilation ray w.r.t. λ based at X is a Thurston geodesic.

Further, the goal is to show that this is true for all λ which are limits of simple closed geodesics.

References

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William P. Thurston, *Minimal stretch maps between hyperbolic surfaces*, 1998.