On the geodesics of Thurston's Asymmetric Metric

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Certificate of Examination

This is to certify that the dissertation titled **On the geodesics of Thurston's Asymmetric Metric** submitted by **Manvendra Somvanshi** (Reg. No. MS20126) for the partial fulfillment of BS–MS Dual Degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. James Farre at the Max Planck Institute for Mathematics in Science, Leipzig and Prof. Pranab Sardar at the Indian Institute of Science Education and Research, Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Abstract

Thurston's introduction of the asymmetric metric on Teichmuller space brought new interest to the hyperbolic geometric perspective to Teichmuller theory, which earlier was viewed in terms of complex analysis. In this dissertation, I go over several well known results in hyperbolic geometry and Teichmuller theory. Some theorems in Thurston's seminal paper [Thu98] are proved in this dissertation in detail using the proof ideas presented by Thurston. In the last chapter of this dissertation, the dilation ray construction in [CF21] is shown to be a Thurston geodesic for the case when the geodesic lamination is a pants decomposition.

List of Figures

Contents

Introduction

The Teichmüller space, Teich(S), was mostly studied to understand complex analytic structures that a given surface S can admit and maps between them. Teichmüller had introduced a metric on Teich(S) which measured the optimal quasi-conformal constant of quasi-conformal homeomorphisms between two complex structures on S. A detailed review of this is given in [FM11].

Thurston's motivation for introducing the asymmetric metric in [Thu98] was to give a geometric perspective on Teich(S). For compact surfaces with negative Euler characteristic there is a bijective correspondence between complex structures and hyperbolic structures. Thus by replacing quasi-conformal maps between complex structures with Lipschitz maps between hyperbolic structures, we obtain Thurston's asymmetric metric as described in chapter II. Constructing Lipschitz maps with the optimal Lipschitz constant between hyperbolic structures is equivalent to constructing geodesics for Thurston's metric. In [Thu98], the construction of stretch rays is the first construction of geodesic rays on Thurston's metric. Since then many more constructions for geodesics in the asymmetric have been given, for example by Papadopolous, Yamada, Therét. See [PS24] and references there in for an overview of different constructions of geodesics in Thurston's Asymmetric metric. In section 2 of chapter II of this thesis key properties of Thurston's metric are proved in detail, filling all the gaps in the proofs presented in [Thu98].

In [Luo07], Luo shows that by cutting up a surface with compact boundary Σ into colored right-angled hexagons and assigning to each non-boundary edge e of the hexagons the radius coordinate $z(e)$, the map from Teich(Σ) to the space of weighted filling arc systems, $|\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)| \times \mathbb{R}_+$, defined as $Y \mapsto \sum_e z(e)e$ is a homeomorphism. More recently this result has been generalized using orthogeodesic foliations and spines of surfaces; which are described in chapter III. Given any hyperbolic surface with boundaries and crowns Y the closest point projection to the boundary of Y defines a measured foliation on Y called the orthogeodesic foliation $\mathcal{O}(Y, \partial Y)$. Transverse to the core of the spine there are special arcs called "dual arcs" arcs to the spine which form a filling arc system on Y. In [CF21], using the measure of the edges of the spine corresponding to the dual arcs as weights the map $\underline{A} : \mathrm{Teich}(\Sigma) \to |\mathscr{A}_{\mathrm{fill}}(\Sigma, \partial\Sigma)| \times \mathbb{R}_+$ is defined, and is shown to be a homeomorphism. This map is actually equivalent to the map described by Luo when the surface has compact boundary.

Given a closed hyperbolic surface X and a lamination λ on X Calderon and Farre define dilation rays based at X with respect to λ in $\mathrm{Teich}(S)$. This construction is defined in section 3 of chapter III of this dissertation. In their paper, Calderon and Farre ask whether these dilation rays are geodesics in Thurston's metric. In the final section I present a new result proving that whenever λ is a pants decomposition of X then dilation rays are geodesics in Thurston's metric (theorem III.15).

The key idea in the proof is to "average" Thurston stretch maps along "opposite" maximal laminations so that the dual arcs of the pants are fixed at the endpoints. To show that this average map fixes the endpoints of the dual arcs a new space $\mathrm{Teich}_\partial(Y)$ is constructed which forms an $\mathbb{R}^n-$principal bundle over $\mathrm{Teich}(\Sigma)$ where n is the number of boundary components of Y. Then it is shown that there is a special section of this where the dual arcs are preserved and that the points of this section are marked by the average map.

The first chapter of this dissertation covers some basic hyperbolic geometry and highlights some important results and concepts. The second chapter covers some Teichmuller theory and Thurston metric. The final chapter sets up the background for Dilation rays and the final section is devoted to proving theorem III.15.

Chapter I

Hyperbolic Geometry

This chapter is a quick review of Hyperbolic geometry and Hyperbolic surfaces. Most results mentioned here are well known and thus proofs are omitted. Most of the proofs can be found in [Kat92, CB88, Bus10].

I.1. The Plane, Disk, and Hyperboloid Model

This section introduces three models of the hyperbolic plane and some of it's basic properties. These three will be used in later sections based on convenience in proofs.

Definition I.1. The Hyperbolic plane is the upper-half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ with the Riemannian metric $\mathrm{d}s^2 = \left(\mathrm{d}x^2 + \mathrm{d}y^2\right)/y^2$.

This means that in the tangent space $T_z\mathbb{H}^2$ we have the inner product

$$\langle \xi, \eta \rangle = \frac{\Re(\xi \cdot \overline{\eta})}{\Im(z)^2}. \tag{I.1}$$

This inner product induces the norm $\|\cdot\|$ on $T_z\mathbb{H}^2$. Using this Riemannian metric we can define the distance on $\mathbb{H}^2$ between two points z, w as follows:

$$d(z, w) = \inf_\gamma \int_\gamma \|\dot{\gamma}(t)\| \mathrm{d}t = \inf_\gamma \int_\gamma \frac{|\dot{\gamma}(t)|}{\Im(\gamma(t))} \mathrm{d}t. \tag{I.2}$$

where the infimum is over all curves γ between z and w. The group of Mobius transformations with real coefficients are isometries of $\mathbb{H}^2$. This group is generated by $z \mapsto z + a$, $z \mapsto \lambda z$, and $z \mapsto -1/z$. It is a straightforward computation to show that this group acts transitively on $\mathbb{H}^2$. Mobius maps with real coefficients form a group isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Hence we view $\mathrm{PSL}(2, \mathbb{R})$ as a subgroup of $\mathrm{Isom}(\mathbb{H}^2)$. In fact, we know a much stronger result:

Proposition I.2 (see [Kat92, CB88]). *The isometry group of $\mathbb{H}^2$ is generated by $PSL(2,\mathbb{R})$ along with the map $z \mapsto 1/\bar{z}$. The orientation preserving isometries are precisely the elements of $PSL(2,\mathbb{R})$.*

In any metric space X, we define a curve $\gamma : [a,b] \to X$ to be a *geodesic* if γ is an isometry, i.e. $d(\gamma(t),\gamma(t')) = |t-t'|$ for all $t,t' \in [a,b]$. The geodesics of $\mathbb{H}^2$ can be completely classified:

Proposition I.3 (see [Kat92, CB88]). *$\mathbb{H}^2$ is uniquely geodesic and the geodesics of $\mathbb{H}^2$ are semi-circles and lines which are perpendicular to the boundary $\partial\mathbb{H}^2$.*

Remark. Note that the boundary $\partial\mathbb{H}^2$ is the boundary of the upper half plane inside the Riemann sphere. Thus $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$.

Let $D(0,1)$ be the open unit disk in $\mathbb{C}$. There is a natural biholomorphism $\varphi : \mathbb{H}^2 \to D(0,1)$ given by $z \mapsto (z-i)/(z+i)$. Using this map we can pull-back the Riemannian metric of $\mathbb{H}^2$ to $D(0,1)$. This metric on $D(0,1)$ turns out to be

$$\mathrm{d}s^2 = \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2} \tag{I.3}$$

This is called the *Poincare Disk model* of the hyperbolic plane. Using the biholomorphism φ, we deduce that the geodesics in the disk with the above metric are diameters and circular arcs which are perpendicular to the boundary, which is the unit circle in $\mathbb{C}$. We denote this model by $\mathbb{H}^2$ as well, as in most cases it either will not matter which model we are referring to or it will be clear by context. The isometries in the disk model explicitly are given by maps of the form

$$z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z}, \ |a| < 1 \tag{I.4}$$

These form a group isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

Consider $\mathbb{R}^3$ with the bilinear form $h(X,Y) = x_1y_1 + x_2y_2 - x_3y_3$ and define the set $H = \{X \in \mathbb{R}^3 \mid h(X,X) = -1 \ \& \ x_3 > 0\}$. The bilinear form induces a Riemannian metric on H given by $\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2$. The space H with this induced metric is isometric to $\mathbb{H}^2$. This can be seen explicitly via the map $f : D(0,1) \to H$

$$f(r,\theta) = \left(\frac{2r}{1-r^2}\cos(\theta), \frac{2r}{1-r^2}\sin(\theta), \frac{1+r^2}{1-r^2}\right) \tag{I.5}$$

Thus H is just another model for $\mathbb{H}^2$ and is called the *Hyperboloid model*. The isometries in this model are just matrices $A \in GL_3(\mathbb{R})$ which preserve the bilinear form. This is the group $SO_0(2,1)$, which is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. Explicitly these are generated by the matrices of the form (see [Bus10]):

$$L_\sigma = \begin{pmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\&\; M_\rho = \begin{pmatrix} \cosh\rho & 0 & \sinh\rho \\ 0 & 1 & 0 \\ \sinh\rho & 0 & \cosh\rho \end{pmatrix} \tag{I.6}$$

Consider the point $p_0 = (0, 0, 1)$ and the curve $\mu(t) = M_t(p_0)$, which by definition is a geodesic. This geodesic corresponds to the geodesic ie^t in the plane model.

I.2. Area and Trigonometry

A polygon in $\mathbb{H}^2$ is a subset $P \subset \mathbb{H}^2 \cup \partial\mathbb{H}^2$ which is bounded by finitely many piece-wise by geodesics. Studying polygons gives a concrete picture of geometry in $\mathbb{H}^2$. The key aspects to understand are what is the area of a given polygon and how are the sides and angles of the polygons related.

Definition I.4. The area measure induced by the Riemannian metric on the plane $\mathbb{H}^2$ is:

$$\mu(S) = \int_S \frac{\mathrm{d}x\mathrm{d}y}{y^2} \tag{I.7}$$

where $S \subset \mathbb{H}^2$.

This measure is preserved under the action of $\mathrm{PSL}(2, \mathbb{R})$. The most important result regarding the area of polygons is the Gauss-Bonnet theorem.

Theorem I.5 ([Kat92])**.** *The area of a hyperbolic triangle Δ with angles α, β, γ is $\mu(\Delta) = \pi - \alpha - \beta - \gamma$.*

An immediate corollary of this is that the area of any $n-$sided polygon P with angle $(\alpha_i)_{i=1}^n$ is $\mu(P) = (n-2)\pi - \sum_i \alpha_i$.

Given a triangle Δ with side lengths a, b, c and angles opposite to the sides α, β, γ, is it possible to deduce a relation between these quantities like in the case of Euclidean geometry? The answer is yes!

Theorem I.6 (Sine and Cosine Rule)**.** *Given a triangle Δ with side lengths a, b, c and angles α, β, γ:*

$$\cosh c = -\sinh a \sinh b \cos\gamma + \cosh a \cosh b \tag{I.8}$$

$$\cos\gamma = \sin\alpha \sin\beta \cosh c - \cos\alpha \cos\beta \tag{I.9}$$

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma} \tag{I.10}$$

Proof from [Bus10]. We work in the Hyperboloid model. Without loss of generality assume that the side c is placed along the geodesic μ and the vertex with angle α is at p_0. Note that M_ρ performs a translation along μ and L_σ performs a rotation about p_0. From figure I.1 we get,

$$L_{\pi-\alpha} M_b L_{\pi-\gamma} M_a L_{\pi-\beta} M_c = I_3 \tag{I.11}$$

Carrying out the matrix multiplication and rearranging we complete the proof. □

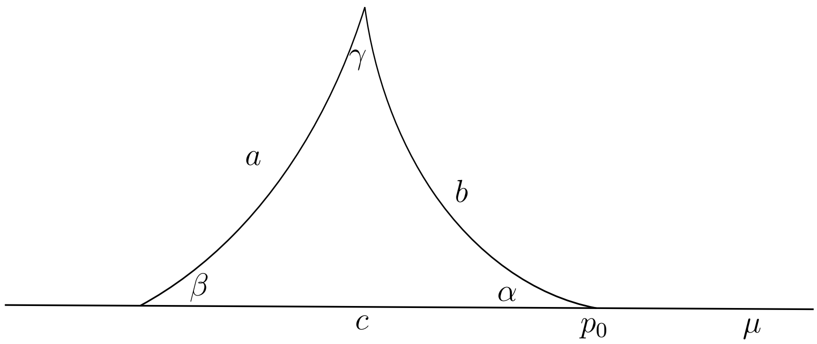

Figure I.1: The translation M_c brings the vertex with angle β to p_0 and $L_{\pi-\beta}$ rotates it so that the length a is along μ. Repeating this, we eventually get the triangle back.

The same technique can be applied to any polygon to get relations between the sides and angles. As right angled hexagons will be important later on I state the Sine and Cosine rules for that here.

Theorem I.7 (Sine and Cosine Rule for Hexagons). *Let H be a hexagon with side lengths $a, \gamma, b, \alpha, c, \beta$ (in that order). Then*

$$\cosh c = \sinh a \sinh b \cosh \gamma - \cosh a \cosh b \tag{I.12}$$

$$\coth \alpha \sinh \gamma = \cosh \gamma \cosh b - \coth a \sinh b \tag{I.13}$$

$$\frac{\sinh a}{\sinh \alpha} = \frac{\sinh b}{\sinh \beta} = \frac{\sinh c}{\sinh \gamma} \tag{I.14}$$

I.3. (X, G) structures and Hyperbolic Surfaces

Let X be a differential $n-$manifold and let G be a Lie group which acts analytically[1] and transitively on X via diffeomorphisms. A smooth map $f : U \to X$ where $U \subset X$ is said to be locally-G if for all open sets $U_i \subset U$ there exists a $g_i \in G$ such that $g_i|_{U_i} = f|_{U_i}$.

[1]This means that whenever $g_1, g_2 \in G$ such that for some open set $U \subset X$, $g_1|_U = g_2|_U$ then $g_1 = g_2$. The motivation for this comes from analytic continuation.

Definition I.8 ((X, G)−structures as defined in [Gol22])**.** We say that an n−manifold M has (X, G) structure if there is an atlas $\{(U_\alpha, \varphi_\alpha : U_\alpha \to X)\}$ of M where φ_α are homeomorphisms onto their images and the transition maps are locally-G maps.

A map $f : M \to N$ between two (X, G)−manifolds is called a (X, G)−map if $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is a locally-G for coordinate patches (U_α, φ_a) on M and (V_β, ψ_β) on N. It is easy to check that any locally-G map f has the unique extension property, i.e. there is a unique $g \in G$ such that $g|_U = f$.

Given a (X, G)−manifold M there is a unique (X, G)−structure on the universal cover $p : \tilde{M} \to M$. Moreover, the Deck group $\pi_1(M)$ will act on $\tilde{M}$ by (X, G)−automorphisms. Fix a base point $p \in \tilde{M}$ and a coordinate neighborhood (V, ψ) of p. Define an extension $\mathrm{Dev}_{p,\psi} : \tilde{M} \to X$ of ψ as follows:

1. For any point $m \in \tilde{M}$ let $\gamma : [0, 1] \to \tilde{M}$ be a path from p to m. The path can be covered by finitely many coordinate charts $\{V_i, \psi_i\}$ such that $V_0 = V$ and $\psi_0 = \psi$ and there are intervals (a_i, b_i) with

$$0 = a_0 < a_1 < b_0 < a_2 < b_1 < a_3 \cdots < b_n = 1 \tag{I.15}$$

   and $\{\gamma(t)\}_{a_i<t<b_i} \subset U_i$. See figure I.2.

2. By the unique extension property, there is a $g_i \in G$ such that $g_i \circ \psi_i = \psi_{i-1}$ on the open set $U_i \cap U_{i-1}$.

3. Define $\mathrm{Dev}_{p,\psi} = g_1 \cdots g_n \psi_n(m)$.

The map $\mathrm{Dev}_{p,\psi} : \tilde{M} \to X$ is well defined and unique extension of $\psi : V \to X$. For some other chart (V', ψ') the extension will differ by the action of some unique g. For some $\gamma \in \pi_1(M)$ both Dev and Devγ are developing maps extending different charts and thus it follows that there is some unique $g \in G$ such that $\mathrm{Dev}\gamma = g\mathrm{Dev}$. The map hol : $\gamma \mapsto g$ is an injective homomorphism from $\pi_1(M) \to G$. A proof of these claims can be found in [Rat94]. The map Dev is called the *developing map* and hol is called the *holonomy*.

The following theorem from [Gol22] summarizes the above discussion:

Theorem I.9 (Development Theorem)**.** *Let M be an (X, G)−manifold and $p : \tilde{M} \to M$ be a universal cover of M. Then there exists a pair ($Dev : \tilde{M} \to X, hol : \pi_1(M) \to$*

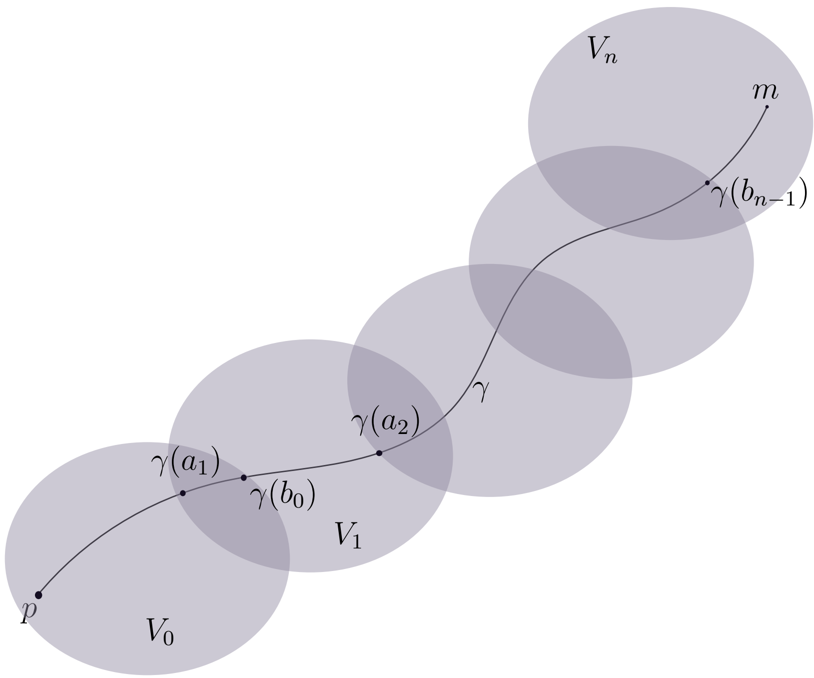

Figure I.2: The construction of developing map.

G) such that for all $\gamma \in \pi_1(M)$ the diagram

$$\begin{tikzcd} \tilde{M} \arrow[r, "Dev"] \arrow[d, "\gamma"'] & X \arrow[d, "hol(\gamma)"] \\ \tilde{M} \arrow[r, "Dev"'] & X \end{tikzcd}$$

commutes. Here Dev is a developing map and the homomorphism hol is called the holonomy representation of $\pi_1(M)$. Moreover, if there is another pair (Dev', hol') with the same properties, then $Dev' = gDev$ and $hol' = g \cdot hol \cdot g^{-1}$ for some unique $g \in G$.

Definition I.10. A hyperbolic surface M is a $(\mathbb{H}^2, \mathrm{PSL}(2, \mathbb{R}))-$manifold. The (X, G) maps in this case are just local isometries.

From the development theorem, $\pi_1(M)$ can be viewed as a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ using holonomy and we also have a developing map Dev : $\tilde{M} \to \mathbb{H}^2$. The local $\mathbb{H}^2$ structure gives the manifold M a Riemannian metric, and geodesics in M will locally be mapped to geodesics in $\mathbb{H}^2$. A hyperbolic surface M is said to be *complete* if every geodesic in M can be extended indefinitely.

Theorem I.11. *A simply connected and complete hyperbolic surface M is isometric to $\mathbb{H}^2$.*

Proof from [CB88]. The proof involves constructing the map $\exp : \mathbb{H}^2 \to M$ for a fixed $p \in M$ and a chart (V, ψ) around p follows: for any point $z \in \mathbb{H}^2$ let γ be the geodesic from $\psi(p)$ to z, and pull back the geodesic arc $\gamma \cap \psi(V)$ to a geodesic γ' in M. Define $\exp(z)$ to be the point on γ' such that its distance from p is $d(z, \psi(p))$. The inverse of this map is the developing map extending ψ. Since both of these maps are local isometries, it follows that they are isometries in this case. For details see [CB88]. $\square$

Since any closed surface X is complete it's universal cover is complete thus, by the above theorem, $\tilde{X}$ is isometric to $\mathbb{H}^2$. Rephrasing, any closed hyperbolic surface X is isometric to $\mathbb{H}^2/\Gamma_X$ where Γ_X is some discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Motivated by this discussion the concept of fundamental domains can be defined.

Definition I.12. Let Γ be a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. A domain $F \subset \mathbb{H}^2$ is said to be a *fundamental domain* of Γ if

1. It tiles $\mathbb{H}^2$, i.e. $\bigcup_{\gamma \in \Gamma} \gamma F = \mathbb{H}^2$,
2. $F^\circ \cap \gamma F^\circ = \emptyset$ for all $\gamma \in \Gamma$.

Although the fundamental domain is not unique often one uses a *Dirichlet Domain* construction for the fundamental domain of the group, see [Kat92]. The Dirichlet domain of closed surfaces is a hyperbolic polygon with finitely many sides. The quotient $\mathbb{H}^2/\Gamma$ can be thought of as gluing the sides of the Dirichlet domain of Γ in pairs. Moreover, these side pairings generate the group Γ ([Kat92]).

For the sake of later discussions define an affine manifold as follows.

Definition I.13. An affine manifold M is a $(\mathbb{R}^n, \mathrm{Aff}(\mathbb{R}^n))-$manifold. The $(X, G)-$maps in this case are called *affine* maps.

Here are two examples of affine manifolds which is all that is really needed for the purposes of this thesis.

Example I.14. Consider the positive real line $\mathbb{R}^+$ with the usual manifold structure. The map $\log : \mathbb{R}^+ \to \mathbb{R}$ which maps $y \mapsto \log(y)$ is a homeomorphism with inverse being $x \mapsto e^x$. This map gives an affine manifold structure to $\mathbb{R}^+$ with local charts given by restrictions of log. Using this structure it is easy to see that any affine map $f : \mathbb{R}^+ \to \mathbb{R}^+$ is of the form $y \mapsto by^a$. With this structure $\mathbb{R}^+$ is affine isomorphic to $\mathbb{R}^+$.

Example I.15. Another example of an affine manifold is a circle S^1. Circle has a natural affine structure as it can be written as the quotient $\mathbb{R}/a\mathbb{Z}$. An affine isomorphism between two affine circles $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/a\mathbb{Z}$ is such that the lift $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is equivariant affine map, i.e. it's lift is of the form $\tilde{f}(x) = ax + b$ for any $b \in \mathbb{R}$ in this example.

I.4. Foliations, Geodesic Laminations, and Transverse Measures

The theory of foliations has many important applications in the study of Teichmuller spaces, see [FLP21]. In this section some basic facts about measured foliations and laminations are presented.

Definition I.16. A measured foliation $\mathscr{F}$ on a compact surface S with singularities of order $k_1, \cdots, k_n \in \mathbb{N}$ at points $x_1, \cdots, x_n$ is given by an open cover $\{U_i\}$ of $M - \{x_1, \cdots, x_n\}$ and a non-vanishing, smooth, closed, and real valued one-form ω_i on each U_i such that:

1. $\omega_i = \pm\omega_j$ on $U_i \cap U_j$
2. At each x_i there is a local chart $(u, v) : V \to \mathbb{R}^2$ such that $z = u + iv$ with $\omega_i = \Im(z^{k_i/2}\mathrm{d}z)$ on $V \cap U_i$, for some branch of $z^{k_i/2}$.

$\{(U_i, \omega_i)\}$ is called the atlas of the foliation.

This definition has a nice geometrical picture. Suppose $p \in S - \{x_1, \cdots, x_n\}$ lies in U_i and let $v_i : U \to \mathrm{T}(U)$ be a vector field which is in the kernel of ω_i. The unoriented flow lines of v_i and v_j in $U_i \cap U_j$ due to the compatibility condition of the one forms, and thus they join up to give "lines" on the surface which are called the *leaves* of the foliation $\mathscr{F}$. By definition, the leaves are disjoint and cover the entire surface except for the singularities x_i where $k_i + 2$ leaves meet. The "measure" part of measure foliation comes from the following: if α is an arc that is transverse to the flow lines of v_i then the integral of $|\omega_i|$ on the arc $U_i \cap \alpha$ gives a measure μ_α to the arc α which is well defined because of the compatibility condition 1. This measure is called the *transverse measure of the foliation.* This measure has full support, meaning that every point in $S - \{x_1, \cdots, x_n\}$ is in the support of μ_α for some α.

The following are some examples of *measured foliations.*

Example I.17. Consider the foliation $\mathscr{F}$ on $\mathbb{R}^2$ with singularity at $(0,0)$ and chart $(U = \mathbb{R} - (0,0), \omega = ydx + xdy)$. A vector field in the kernel of this one form is $x\partial_x - y\partial_y$, and thus the flow lines corresponding to this are $\Phi(t,(x,y)) = (xe^t, ye^{-t})$. In any neighborhood V of the origin consider the natural chart inclusion chart. Then in $V \cap U$ we have $\omega = \Im(z\mathrm{d}z)$, implying that the singularity is of order 2. Thus geometrically the foliation $\mathscr{F}$ looks as in I.3.

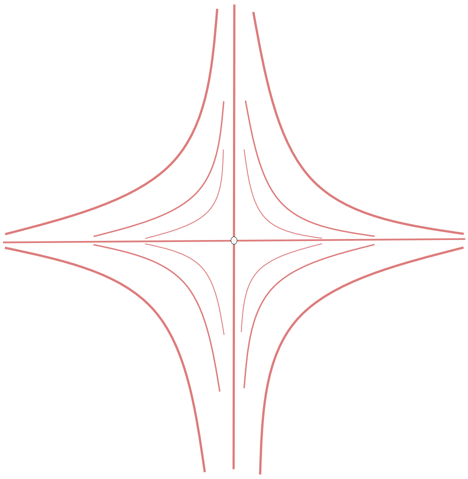

Figure I.3: The measured foliation $\mathscr{F}$ on $\mathbb{R}^2$ constructed in example I.17.

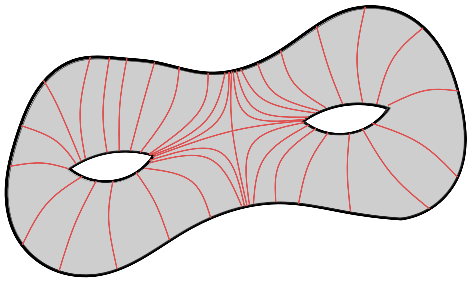

Figure I.4: The measured foliation $\mathscr{F}$ on S_2 constructed using a quadratic differential with 2 zeros of order 2.

Example I.18. Let M be any compact Riemann surface and q be a holomorphic quadratic differential on M which has zeros $x_1, \cdots, x_n$ of order $k_1, \cdots, k_n$. This means that in local charts q takes the form $f(z)\mathrm{d}z^2$ where f is holomorphic. Consider $M - \{x_1, \cdots, x_n\}$ and an cover $\{U_i\}$ where U_i is simply connected. In each cover define $\omega_i = \Im(\sqrt{q})$, where a choice of the branch is made in the charts. By definition, ω_i are compatible. For any neighborhood V of x_i there are charts given by integrating ω_i such that locally q is $z^{k_1}\mathrm{d}z^2$. Thus both conditions are satisfied and $\{U_i, \omega_i\}$ is a measured foliation $\mathscr{F}$ on M.

This is an important example since it was shown in [HM79] that almost all measured foliations on Riemann surfaces are of the above type, i.e. they are determined by the imaginary foliation of the quadratic differential.

Foliations are very closely related with the topology of the space. Using the Poincare-Hopf formula, the following result for foliations can be proven (see [FLP21]).

Theorem I.19 (Euler-Poincare Formula). *If $\mathscr{F}$ is a measured foliation with singularities $x_1, \cdots, x_n$ of order $k_1, \cdots, k_n$ on a compact surface S such that no singularity lies on the boundary of S. Then*

$$2\chi(S) = -\sum_{i=1}^{n} k_i \tag{I.16}$$

This significantly restricts the measured foliations a surface S can admit. For instance, in S_2 there can only be three types of measured foliations: with singularities of order $(1,1,1,1)$, $(1,1,2)$, $(2,2)$. Figure I.4 is a foliation of the type $(2,2)$.

Since small perturbations of the leaves of a foliation are not geometrically relevant and so an equivalence relation is introduced as follows. On a surface S the foliations $\mathscr{F}_1 \sim \mathscr{F}_2$ if

- there is an isotopy of S which transforms $\mathscr{F}_1$ to $\mathscr{F}_2$
- $\mathscr{F}_1$ can be transformed into $\mathscr{F}_2$ by finitely many transformations of the following type:

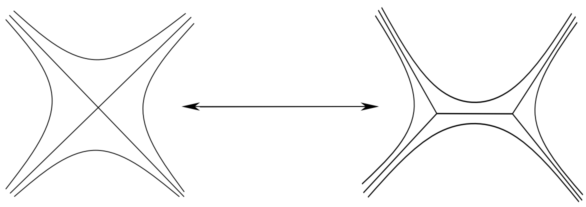

Figure I.5: Whitehead moves.

Transformations as in I.5 are called *Whitehead moves.* The collection of all measured foliation on S upto this equivalence is denoted $\mathcal{MF}(S)$, the space of measured foliations.

Working with measured foliations can get quite complicated as they are often thought of as the equivalence classes in $\mathcal{MF}(S)$. In the case of hyperbolic surfaces, there is a very simple and well understood geometric object called *geodesic lamination* which encodes equivalent information about our surface S as a foliation.

Definition I.20. A geodesic lamination λ on a hyperbolic surface M is a collection of pairwise disjoint complete simple geodesics $\{\gamma_\alpha\}$ such that the $\cup_\alpha \gamma_\alpha$ is a closed set in M.

Examples of geodesic laminations are closure of any disjoint collection of geodesics in the surface. This forms a lamination since under the Hausdorff metric geodesics

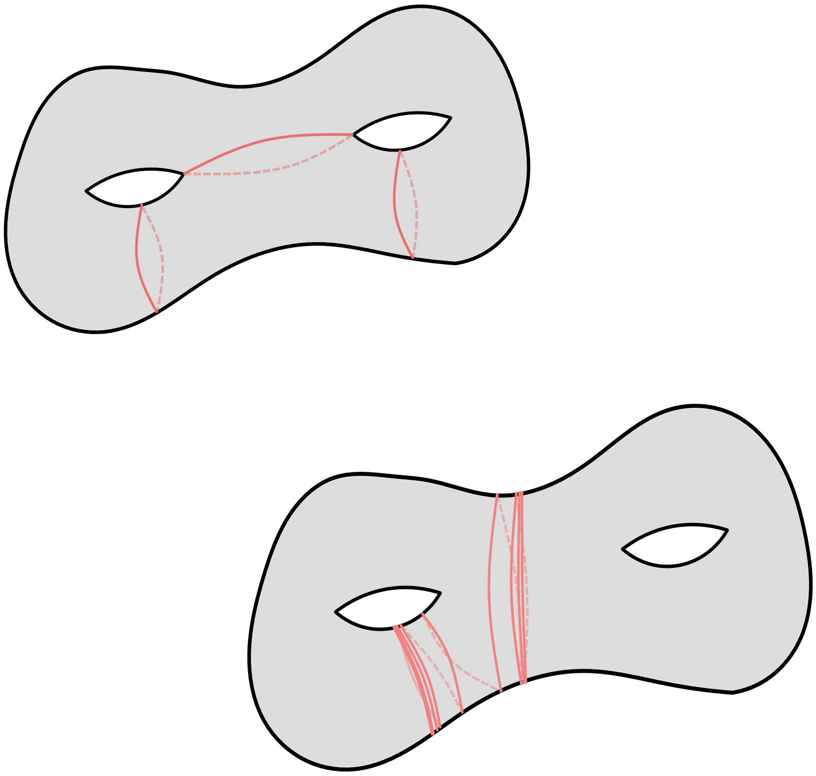

Figure I.6: The first image is a measured geodesic lamination with the transverse measure just being the counting measure. The second image shows a lamination which is the closure of a geodesic spiralling onto two compact geodesics. Here the compact geodesics are proper but not isolated.

converge to geodesics (see [CB88]). The leaf γ is said to be *isolated* if at each point $x \in \gamma$ there is a small neighborhood U such that $(U, U \cap \lambda)$ is homeomorphic to $(\text{Disk}, \text{diameter})$. A leaf γ is called *proper* if for every point $x \in \gamma$ there is a neighborhood U such that $U \cap \gamma$ has only one connected component. It is easy to see that every isolated leaf is a proper leaf. A *minimal sub-lamination* $\lambda' \subset \lambda$ is a lamination which does not contain any sub-lamination. A geodesic lamination λ is said to be maximal if there is no other geodesic lamination on M containing λ.

Remark. Geodesic laminations on hyperbolic surfaces are measure zero subsets of the surface [CB88]. This can be used to see that every maximal lamination cuts the surface M into hyperbolic triangles: if not then one can always add another simple geodesic on the component which is not a triangle.

Definition I.21. A transverse measure on a geodesic lamination λ is an assignment μ which assigns to each transverse arc α a Radon measure μ_α such that:

1. if $\beta \subset \alpha$ then $\mu_\beta = \mu_\alpha|_\beta$.

2. if α is homotopic to α' relative to λ then the measure of any measurable subset of α is invariant under the homotopy.

We say that the transverse measure has full support when $\mathrm{supp}(\mu_\alpha) = \lambda \cap \alpha$ for all transverse arcs α.

Remark. A leaf γ is said to be in the support of the transverse measure if $\gamma \subset \bigcup_\alpha \mathrm{supp}(\mu_\alpha)$. In the case when the measure is of full support every leaf is in the lamination.

A theorem of Thurston ([Thu22]) states that every geodesic lamination admits a transverse measure (may not be of full measure). The idea behind the proof of this is to approximate a transverse measure using counting measure. A geodesic lamination with a transverse measure is called a measured geodesic lamination. The following theorem by Levitt proves some important properties of leaves of a lamination.

Theorem I.22 (Levitt [Lev83])**.** *Let λ be a geodesic lamination on a closed hyperbolic surface M and $p : \tilde{M} \to M$ be the universal cover of M. Let $\gamma \subset \lambda$ be a leaf of the lamination. Then the following are equivalent:*

1. *γ is isolated*
2. *Each geodesic in $p^{-1}(\gamma)$ is an isolated point in $G_\lambda = p^{-1}(\lambda) \subset \tilde{M}$.*
3. *The complement of γ in λ is compact.*

If γ is non-compact then 1,2,3 are also equivalent to

5. *γ is proper*
6. *γ belongs to no minimal sublamination of λ*
7. *No transverse measure of λ contains γ in its support.*

Moreover, there can only be finitely many isolated leaves in a lamination; the other leaves are partitioned into finitely many minimal sublaminations.

If λ is a measured geodesic lamination of full support then, as a consequence of the above theorem, it cannot have any leaves which are spiraling onto compact leaves as they are isolated and thus cannot be in the support. A non-compact leaf γ which does not spiral onto a compact leaf belongs to the minimal sublamination $\overline{\gamma}$ (closure of the leaf in M).

The space of all measured laminations with full support is denoted $\mathcal{ML}(M)$. Then the following theorem is due to Thurston:

Theorem I.23. *The space of measured foliations on M is homeomorphic to the space of measured laminations on M.*

Remark. The theorem above claims that $\mathcal{MF}(M)$ and $\mathcal{ML}(M)$ are homeomorphic but I have not defined a topology on these spaces. This is because defining the topology on these takes some work and it will not be relevant to the remainder of this thesis. For a more rigorous treatment of this see [Thu22, FLP21].

Remark. In [Thu22], a proof of the above theorem is given using *train tracks.* A more general study of the relationship between geodesic laminations and foliations (both measured and non-measured) can be found in [Lev83].

I.5. Maps between Hyperbolic Surfaces

This section is devoted to understanding and highlighting some important properties of homotopy equivalence between hyperbolic surfaces. Throughout this section let X, Y be compact hyperbolic surfaces which are homeomorphic to a topological surface S, $p_X : \tilde{X} \to X$ & $p_Y : \tilde{Y} \to Y$ be their universal covers, and $f : X \to Y$ be a homotopy equivalence. Let $f_* : \pi_1(X) \to \pi_1(Y)$ be the induced isomorphism on the fundamental group. Denote by Γ_X and Γ_Y some holonomy representation of $\pi_1(X)$ and $\pi_1(Y)$ in $PSL_2(\mathbb{R})$. Whenever X, Y are assumed to be closed their universal covers are identified with $\mathbb{H}^2$.

Proposition I.24. *Let $f : X \to Y$ be a homotopy equivalence. Then any lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a homotopy equivalence and is $(\pi_1(X), \pi_1(Y))-$equivariant: for each $\alpha \in \pi_1(X)$ we have $f_*(\alpha)\tilde{f} = \tilde{f}\alpha$.*

This follows directly from basic results in algebraic topology. When X, Y are closed we want to understand when $\tilde{f}$ can be extended to the boundary $\partial\mathbb{H}^2$. The following theorem gives us an answer.

Theorem I.25. *Let $f : X \to Y$ be a homotopy equivalence between closed hyperbolic surfaces. Then any lift $\tilde{f}$ of f has a unique equivariant extension $\bar{f}$ on $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ such that the restriction of this map to the boundary $\partial f : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ is a homeomorphism invariant under homotopies of f.*

Proof Idea. By Milnor-Schwarz lemma, the orbit maps of the fundamental group of X, Y at $0 \in \mathbb{H}^2$, $q_X : \Gamma_X \to \mathbb{H}^2$ and $q_Y : \Gamma_Y \to \mathbb{H}^2$, are quasi-isometries with respect to the word metric and f_* is bi-Lipschitz. Due to equivariance of $\tilde{f}$ the diagram

$$\begin{tikzcd}
\Gamma_X \arrow[r, "f_*"] \arrow[d, "q_X"'] & \Gamma_Y \arrow[d, "q_Y"] \\
\mathbb{H}^2 \arrow[r, "\tilde{f}"'] & \mathbb{H}^2
\end{tikzcd}$$

commutes. This implies that $\tilde{f}$ is a quasi-isometry on $\mathbb{H}^2$, and thus it maps geodesics to quasi-geodesics. For any point $x \in \partial\mathbb{H}^2$ let $\partial f(x)$ be the end-point of the quasi-geodesic $f(\alpha)$ where α is the extension of the geodesic $[0, x]$. Since quasi-geodesics are always a bounded distance away from geodesics (using stability of quasi-geodesic theorem in [BH13]), this endpoint always exists. The remaining properties can be proven without much difficulty. □

Remark. The proof of this theorem in [CB88] does not use Milnor-Schwarz lemma and arrives at the same conclusion using elementary analysis and geometry. Although, using the same idea as in the proof presented above this theorem can also be generalized to surfaces with boundary by replacing $\partial\mathbb{H}^2$ with the Gromov boundary $\partial\tilde{X}$ and $\partial\tilde{Y}$.

The following is a direct Corollary of I.25 for closed surfaces, but I have stated it in a way so that it works for surfaces with boundaries as well.

Proposition I.26. *Let X be a compact hyperbolic surface (with or without boundary) and $I : X \to X$ be some isometry which is isotopic to the identity. Then there is a lift $\tilde{I} : \tilde{X} \to \tilde{X}$ of I which is the identity map.*

Proof. Let $H : X \times I \to X$ be an isotopy such that $H_0 = I$ and $H_1 = \mathrm{id}_X$. Then there is a unique lift of H to an isotopy $\tilde{H} : \tilde{X} \times I \to \tilde{X}$ such that $\tilde{H}_1 = \mathrm{id}_{\tilde{X}}$. Let $\tilde{I} = \tilde{H}_0$. Using the fact that $p : \tilde{X} \to X$ is a local isometry and the fact that X is compact, one can conclude that $\tilde{I}$ is an isometry which is isotopic to $\mathrm{id}_{\tilde{X}}$, and moreover it is a bounded distance away from $\mathrm{id}_{\tilde{X}}$. Let $\mathrm{Dev} : \tilde{X} \to \mathbb{H}^2$ be some developing map. Then the map $\mathrm{Dev} \circ \tilde{I}$ is also a developing map. It follows from the discussion in I.3 that there is some $g \in \mathrm{PSL}(2, \mathbb{R})$ such that $\mathrm{Dev} \circ \tilde{I} = g\mathrm{Dev}$. Since $\tilde{I}$ is a bounded distance away from $\mathrm{id}_{\tilde{X}}$ it follows that g is a bounded distance away from $\mathrm{id}_{\mathbb{H}^2}$. Thus it follows that g and $\mathrm{id}_{\mathbb{H}^2}$ agree on $\partial\mathbb{H}^2$, but since any Mobius map is determined by what it does to exactly three points one concludes that $g = \mathrm{id}_{\mathbb{H}^2}$. Hence $\tilde{I}$ is just the identity. □

Corollary. *If $I_1, I_2 : X \to Y$ are isometries of compact hyperbolic surfaces (with or without boundary) such that I_1 is isotopic to I_2 then for any lift $\tilde{I}_1$ of I_1 to the universal cover there is a lift $\tilde{I}_2$ of I_2 such that the two lifts are equal.*

Theorem I.27. *Let $f : X \to Y$ be a homotopy equivalence. Then there is a bijective correspondence between geodesic laminations on X and Y.*

Sketch of Proof. Suppose that λ is a geodesic lamination on X which lifts to a lamination $\tilde{\lambda}$ of $\mathbb{H}^2$. Each leaf of $\tilde{\lambda}$ meets $\partial\mathbb{H}^2$ at two points, say (a, b). Using the boundary map ∂f from I.25, we get two points $(f(a), f(b))$ on $\partial\mathbb{H}^2$, which determines a unique geodesic in $\mathbb{H}^2$. This gives us a collection of disjoint geodesics $\tilde{\lambda}'$ on $\partial\mathbb{H}^2$. This indeed forms a geodesic lamination, which is then pushed down to a geodesic lamination $\lambda' = p_Y(\tilde{\lambda}')$ on Y. □

Remark. In fact, the space of geodesic laminations on X and Y can be proven to be homeomorphic under the topology induced by the Hausdorff metric. The proof of that can be found in [CB88].

Definition I.28. A map $f : X \to Y$ is said to be *Lipschitz continuous* if there is a K such that $d_Y(f(x), f(x')) \leq K d_X(x, x')$ for all $x, x' \in X$. A Lipschitz map f is said to be $K-$Lipschitz if

$$K = \sup_{x \neq x' \in X} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}$$

K is called the Lipschitz constant of f and is sometimes denoted $\mathrm{Lip}(f)$. f is called bi-Lipschitz if it has an inverse which is also Lipschitz.

Proposition I.29. *Every diffeomorphism $f : X \to Y$ is a Lipschitz map with Lipschitz constant $K = \sup_{x \in X} \|df_x\|_{op}$.*

Proof. Let γ be any piece-wise smooth curve on X connecting the points x and x'. Since $\mathrm{d}f : \mathrm{T}^1X \to \mathrm{T}Y$ is smooth, it follows by compactness of X that $x \mapsto \|\mathrm{d}f_x\|_{\mathrm{op}}$ has an upper bound, say K. Then,

$$\begin{aligned}
\text{Length of } f(\gamma) &= \int_0^1 \|\dot{f(\gamma)}(t)\|_Y \mathrm{d}t \\
&= \int_0^1 \|df_{\gamma(t)}(\dot{\gamma}(t))\|_Y \mathrm{d}t \\
&\leq K \int_0^1 \|\dot{\gamma}(t)\|_X \mathrm{d}t \\
&= K \cdot (\text{Length of } \gamma)
\end{aligned}$$

Taking the infimum over all curves γ, we get that f is Lipschitz. Following the same argument for f^{-1} it can be shown that f is bi-Lipschitz. □

Remark. Since every homeomorphism of surfaces is isotopic to a diffeomorphism of those surfaces, it follows that every homeomorphism is isotopic to a bi-Lipschitz map.

Theorem I.30 (Theorem 8.9 in [FM11])**.** *If S is closed surface with genus $g \geq 2$ then every homotopy equivalence $f : S \to S$ is homotopic to a homeomorphism of S.*

Remark. This theorem also applies to compact surfaces with boundaries whenever the homotopy equivalence restricts to a homeomorphism $\partial S \to \partial S$.

Chapter II

Teichmüller Space and Thurston's Metric

In this chapter S represents a closed surface of genus $g \geq 2$, X, Y will be closed hyperbolic surfaces of genus g, and Γ_X, Γ_Y will be faithful representations of their fundamental groups in $\mathrm{PSL}(2, \mathbb{R})$. The goal of this chapter is first to construct the Teichmüller spaces and give some examples and results in the subject. Secondly, it is to understand the Thurston metric on Teichmüller spaces. The proofs presented here of the properties of Thurston metric are all motivated by the ideas of Thurston in [Thu98], but the proofs are presented in detail to fill the gaps in his proofs. This is done since there is a scarcity of sources which present these proofs with all the details.

II.1. Teichmüller Space of Hyperbolic Surfaces

The question which motivates the construction of Teichmüller space is: *what are all the hyperbolic structures which a surface S can admit?* S admits a hyperbolic structure if there is a Riemannian metric on S which locally looks like the metric on $\mathbb{H}^2$. Another way to look at this that S admits a hyperbolic structure if there is a hyperbolic surface X and a diffeomorphism $f : S \to X$. This is equivalent since the diffeomorphism pulls back the metric on X on S. A pair (X, f) where X is a hyperbolic surface and $f : S \to X$ is a homeomorphism is called a *marking* of S. Two markings (X, f) and (Y, g) are said to be equivalent if there is an isometry $I : X \to Y$ such that $If \simeq_{\mathrm{iso}} g$.

Remark. From here on if $f : S \to X$ and $g : S \to Y$ are markings on S and $\varphi : X \to Y$ is some map such that $\varphi f \simeq_{iso} g$ then I will say that "φ is compatible with the marking". Note that for any two other markings $(X', f') \in \mathfrak{X}$ & $(Y' g') \in \mathfrak{Y}$ it follows that φ induces a homeomorphism $\varphi' : X' \to Y'$ which is compatible with these new markings as well.

Definition II.1. The Teichmüller space of S is defined as the space of a hyperbolic markings on S up to the above equivalence.

$$\mathrm{Teich}(S) = \{(X, f : S \to X) \mid f \text{ is orientation preserving}\} / \sim \tag{II.1}$$

We denote the elements of $\mathrm{Teich}(S)$ by $\mathfrak{X}$.

This definition can be extended to surfaces with boundaries with the additional requirement that ∂X is geodesic. From here on whenever X is a hyperbolic surface with boundary it is assumed that the boundary is geodesic.

Remark. Using theorem I.30 the definition of a marking can be weakened to be a homotopy equivalence $f : S \to X$ without any affect to $\mathrm{Teich}(S)$. For surfaces with boundary we need to be a bit more careful. The equivalence relation in the case of surfaces with boundary would then be that $If \simeq g$ such that the homotopy is a through maps which take boundary to boundary and are homeomorphisms on the boundary. Because of this we will sometimes interchange between a marking being a homeomorphism and a homotopy equivalence.[1]

There is another interesting perspective to view $\mathrm{Teich}(S)$, and this is using the holonomy maps and the Development theorem. Suppose that $\mathfrak{X} \in \mathrm{Teich}(S)$ and let (X, f) be some representative of $\mathfrak{X}$. Then $f_* : \pi_1(S) \to \pi_1(X)$ is an isomorphism, and using the developing theorem we have a discrete and faithful representation $\mathrm{hol} : \pi_1(X) \to \mathrm{PSL}(2, \mathbb{R})$. This gives a representation of $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$. If (X', f') is some other representative of $\mathfrak{X}$ then the holonomy representations differ by conjugation by an element of $\mathrm{Isom}(\mathbb{H}^2) \cong \mathrm{PGL}(2, \mathbb{R})$. This leads to the following claim:

Proposition II.2 (see [FM11])**.** *Teich(S) is in bijection with the set of all discrete and faithful representations of $\pi_1(S)$ in PSL$(2, \mathbb{R})$ upto conjugation by PGL$(2, \mathbb{R})$, denoted by DF$(\pi_1(S), PSL(2, \mathbb{R}))/PGL(2, \mathbb{R})$.*

This perspective is interesting as it allows us to directly define a topology on $\mathrm{Teich}(S)$. Give $\pi_1(S)$ the discrete topology and $\mathrm{DF}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$ the compact open topology. Then $\mathrm{DF}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PGL}(2, \mathbb{R})$ can be equipped with the quotient topology which is then pulled back to $\mathrm{Teich}(S)$.

[1] A homotopy equivalence is said to be orientation preserving if the induced isomorphism on the second homology group, which can be identified with $\mathbb{Z}$, maps $1 \mapsto 1$.

Since every nontrivial closed curve γ on a hyperbolic surface X is freely homotopic to a unique closed geodesic (see [CB88]), one can define the length function $\ell_{\mathfrak{X}}$ from the set, $\mathcal{S}$, of all non-trivial closed curves on S to $\mathbb{R}_+$ which maps a closed curve to the length of the unique geodesic in X which is freely isotopic to the curve. This is independent of the choice of representative of $\mathfrak{X}$. This also gives a continuous map $\ell : \text{Teich}(S) \to \mathbb{R}_+^{\mathcal{S}}$ which takes $\mathfrak{X}$ to $\ell_{\mathfrak{X}}$.

Example II.3. Consider the pair of pants, i.e. sphere with three punctures $S_{0,3}$. A hyperbolic pair of pants (with geodesic boundary) can always be cut into two hyperbolic right-angled hexagons, by cutting along the geodesics orthogonal to each pair of boundary curves. By I.7 it follows that exactly three side lengths determine the entire hexagon. Hence the Teichmüller space of $S_{0,3}$ is in bijective correspondence with $\mathbb{R}_+^3$. This map is a homeomorphism since two hexagons with close side lengths are "almost" isometric.

Using the fact that every closed hyperbolic surface X of genus g can be cut along $3g-3$ closed geodesics to give a pairs of pants decomposition of X, and that each of these pants need 3 positive real numbers to determine them exactly, Fenchel and Nielsen arrived at the following theorem.

Theorem II.4 (Fenchel-Nielsen Theorem [FM11])**.** *For any genus g closed surface S, Teich$(S) \cong \mathbb{R}^{-3\chi(S)}$.*

Remark. The extra $3g-3$ parameters come from the fact that when gluing back the pair of pants to recover the genus g surface, the gluing can be done at any angle. Thus for each gluing we have a "twist" parameter. Since there are $3g-3$ curves γ_i in the pants decomposition, the hyperbolic structure is determined by $((\ell_1, \tau_1), \cdots, (\ell_{3g-3}, \tau_{3g-3}))$, where (ℓ_i, τ_i) is the length and twist of γ_i.

II.2. Thurston's Metric on Teichmüller Space

In his paper [Thu98], Thurston asks the following question: given a surface S with two hyperbolic structures $f : S \to X$ and $g : S \to Y$, is there a homeomorphism $\varphi : X \to Y$ compatible with the markings which realizes the least possible value of the Lipschitz constant? In other words if

$$L = \inf_{\substack{\psi : X \to Y \\ \psi f \simeq g}} \text{Lip}(\psi) \tag{II.2}$$

then does there exist a $L-$Lipschitz homeomorphism φ. It turns out that the answer to this question is positive. The definition of the Thurston metric was motivated by this question.

Definition II.5. Let $L : \text{Teich}(S) \times \text{Teich}(S) \to \mathbb{R}_+$ be defined as

$$L(\mathfrak{X}, \mathfrak{Y}) = \inf_{\substack{\psi : X \to Y \\ \psi f \simeq g}} \log(\text{Lip}(\psi)) \tag{II.3}$$

This is called Thurston's asymmetric metric.

Thurston had also defined another metric on $\text{Teich}(S)$ as follows:

Definition II.6. Define $K : \text{Teich}(S) \times \text{Teich}(S) \to \mathbb{R}_+$ as

$$K(\mathfrak{X}, \mathfrak{Y}) = \sup_{c \in \mathcal{S}} \log \left(\frac{\ell_{\mathfrak{Y}}(c)}{\ell_{\mathfrak{X}}(c)} \right) \tag{II.4}$$

This section is mostly devoted to showing that L and K indeed satisfy all the conditions for a metric except symmetry. The symmetry is not really an issue as it can forced by defining a new metric as $L(\mathfrak{X}, \mathfrak{Y}) + L(\mathfrak{Y}, \mathfrak{X})$. In [Thu98] it is also shown that $K = L$, but that will not be required in this thesis. It is easy to see that $K(\mathfrak{X}, \mathfrak{Y}) \leq L(\mathfrak{X}, \mathfrak{Y})$, as a Lipschitz maps stretches any curve by at most it's Lipschitz constant. The other inequality is much harder to prove.

Firstly, it is quite straight forward to check that L, K are not dependent on the choice of representative of $\mathfrak{X}$ and $\mathfrak{Y}$.

Proposition II.7. *Both L, K satisfy the triangle inequality: for any $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \in$ Teich(S),*

$$L(\mathfrak{X}, \mathfrak{Z}) \leq L(\mathfrak{X}, \mathfrak{Y}) + L(\mathfrak{Y}, \mathfrak{Z}) \tag{II.5}$$

$$K(\mathfrak{X}, \mathfrak{Z}) \leq K(\mathfrak{X}, \mathfrak{Y}) + K(\mathfrak{Y}, \mathfrak{Z}) \tag{II.6}$$

Proof. Choose representatives (X, f), (Y, g), and (Z, h) respectively. Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be Lipschitz maps with Lipschitz constants K_1, K_2 respectively. Then $d_Z(\psi\varphi(x), \psi\varphi(x')) \leq K_1 d_Y(\varphi(x), \varphi(x')) \leq K_1 K_2 d_X(x, x')$. Thus $\text{Lip}(\psi\varphi) \leq K_1 K_2 = \text{Lip}(\psi)\text{Lip}(\varphi)$. Since the Lipschitz maps $X \to Z$ which factor through Y forms a strict subset of all Lipschitz maps from $X \to Z$, the triangle inequality for L follows.

Triangle inequality for K just follows just from re-writing:

$$\frac{\ell_Z(c)}{\ell_X(c)} = \frac{\ell_Z(c)}{\ell_Y(c)} \cdot \frac{\ell_Y(c)}{\ell_X(c)}$$

□

Theorem II.8. *$L(\mathfrak{X}, \mathfrak{Y}) \geq 0$ for all $\mathfrak{X}, \mathfrak{Y} \in$ Teich(S). The equality is satisfied if and only $\mathfrak{X} = \mathfrak{Y}$.*

Proof. Suppose that $K = e^{L(\mathfrak{X},\mathfrak{Y})} \leq 1$, and let φ_n be a sequence of Lipschitz homeomorphisms with $\mathrm{Lip}(\varphi_n) \to K$. Since the set of Lipschitz maps with Lipschitz constant bounded by 1 forms an equicontinous family; by Arzela-Ascoli theorem it follows that there is some Lipschitz map $\varphi : X \to Y$ which achieves this constant K (note: φ may not be a homeomorphism, but since X is compact it follows that φ is surjective). From here the proof goes as follows: first it is shown that φ maps disks in X onto disks in Y of the same radius. Using this this it is then shown that φ is an isometry, i.e. $K = 1$.

Denote by μ_X and μ_Y the area measures of X and Y. Using the explicit area formula for hyperbolic disks in $\mathbb{H}^2$ (in [Kat92]) it is easy to show that the area of a small enough disk on any hyperbolic surface is dependent only on it's radius r and since φ contracts distance by K it follows that φ is area non-increasing on closed disks. The outer regularity of the area measure and Lipschitz property of φ also implies that φ maps negligible sets to negligible sets. Suppose that $\mathcal{B} = \{B_n\}$ is a countable family of disjoint closed disks in X such that the union of all these disks has complete measure in X. Using the fact that φ is surjective and maps $X - \bigcup \mathcal{B}$ to a measure zero set,

$$\mu_Y(Y - \varphi(\bigcup \mathcal{B})) = \mu_Y(\varphi(X) - \varphi(\bigcup \mathcal{B})) \leq \mu_Y(\varphi(X - \bigcup \mathcal{B})) = 0.$$

Thus $\varphi(\bigcup \mathcal{B})$ is a set of full measure. If the disk $B_n \in \mathcal{B}$ is of radius $r_n > 0$ then $\varphi(B_n)$ is contained in a disk B'_n of radius r_n since φ has a Lipschitz constant $K \leq 1$. The family $\mathcal{B}' = \{B'_n\}$ is also full measure in Y. Hence,

$$\sum_n \mu_Y\left(B'_n - \varphi(B_n)\right) = \mu_Y(\bigcup \mathcal{B}') - \mu_Y(\varphi(\bigcup \mathcal{B})) = \mu_Y(Y) - \mu_Y(Y) = 0$$

This means that $\varphi(B_n)$ is dense in B'_n but since it is also compact it follows that B_n surjectively maps onto B'_n.

We have just proved that given *any* family of closed disjoint disks of X each disk in the family surjectively maps onto a disk of the same radius. This means that every disk in X maps to a disk of the same radius. Suppose $x \in X$ and let $B(x, r)$ be a disk of radius r centered at x which is surjectively mapped to $B'(y, r)$ in Y. If $\varphi(x)$ is a distance ϵ away from the center of $B'(y, r)$ then the disk $B(x, r - \epsilon/2)$ does not lie inside $B'(y, r)$, a contradiction! Thus φ also maps center to the center of the disk. Moreover, if $x' \in B(x, r)$ lies on the boundary then $\varphi(x')$ also lies on the boundary as concentric disks centered at x map to concentric disks centered at y. For any points x, x' there the disk about x of radius $d_X(x, x')$ gets mapped to a disk of the same

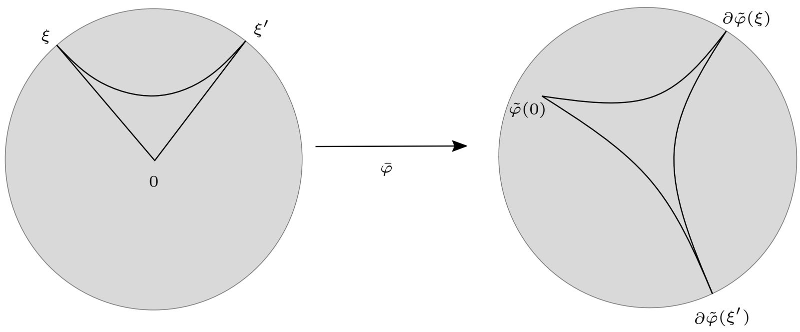

Figure II.1: These represent the triangles Δ and Δ'. The idea in the proof was to use the fact that Gromov product $(x|y)_z$ approximates the length of the perpendicular from z to (x, y).

radius and thus $d_Y(\varphi(x), \varphi(x')) = d_X(x, x')$. Hence φ is an isometry.

Since the homeomorphisms φ_n converge to φ it follows that $\varphi_n \simeq \varphi$ and thus φ is compatible with the marking on X and Y. The second part of the theorem is trivial now. $\square$

Remark. It is not obvious above why a countable family of closed disjoint disk exists in the first place. The answer is given by Vitali's covering lemma and covering theorem. Note that in the construction of the countable disjoint family of closed disks in the proof of Vitali's theorem (see [Eva18]) there is an initial choice of a ball. This has been used in the above proof as well to claim that every disk surjectively maps to a disk of the same radius, as for any given disk one can construct such a family containing this disk.

Theorem II.9. *$K(\mathfrak{X}, \mathfrak{Y}) \geq 0$ for all $\mathfrak{X}, \mathfrak{Y} \in$ Teich(S). The equality is satisfied if and only $\mathfrak{X} = \mathfrak{Y}$.*

To prove this we first prove the following lemma:

Lemma II.10. *Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are distinct elements of Teich(S). If $\varphi : X \to Y$ is a homeomorphism compatible with the markings which lifts to a homeomorphism $\tilde\varphi : \mathbb{H}^2 \to \mathbb{H}^2$. Then the function*

$$D(z) = d(\tilde\varphi(z), \tilde\varphi(0)) - d(z, 0)$$

is unbounded above.

Proof. Suppose that D is the upper of $D(z)$. From the proof of theorem I.25 the lift $\tilde\varphi$ is a quasi-isometric map and thus is maps geodesics to quasi-geodesics and therefore the geodesic $(\bar{\varphi}(z), \bar{\varphi}(w))$ is a bounded distance away from the image of the geodesic (z, w) under $\tilde{\varphi}$, where $z, w \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$. Consider the triangles $\Delta = (0, \xi, \xi')$ and $\Delta' = (\tilde{\varphi}(0), \partial\tilde{\varphi}(\xi), \partial\tilde{\varphi}(\xi'))$ where $\xi, \xi' \in \partial\mathbb{H}^2$, see figure II.1. Thus we have the following inequality involving the Gromov product and the hyperbolicity constant, δ, of $\mathbb{H}^2$,

$$\begin{aligned}(\partial\tilde{\varphi}(\xi)|\partial\tilde{\varphi}(\xi'))_{\tilde{\varphi}(0)} &\leq d(\tilde{\varphi}(0), (\partial\tilde{\varphi}(\xi), \partial\tilde{\varphi}(\xi'))) + \delta \\ &\leq d(0, (\xi, \xi')) + D + \delta \\ &\leq (\xi|\xi')_0 + D + 2\delta\end{aligned}$$

where the first and last inequality follows from the inequality $|(x, y)_z - d(z, (x, y))| \leq \delta$ (see [BH13]), and the second inequality follows from the definition of $D(z)$. Since $e^{-(\xi|\xi')_z}$ is a metric on the boundary for a fixed z the above inequality tells us that $\partial\tilde{\varphi}^{-1}$ is a Lipschitz map on the boundary with respect to the Gromov product induced metrics with fixed point $\tilde{\varphi}(0)$ and 0 respectively on the domain and range (note that these are Lipschitz equivalent metrics, so the fixed point does not really matter).

In this paragraph, we identify $\mathbb{H}^2$ with the upper half plane and $\partial\mathbb{H}^2$ with the extended real line. Since $\partial\tilde{\varphi}^{-1}$ is $K-$Lipschitz for some $K \geq 1$ it is almost everywhere differentiable on $\partial\mathbb{H}^2$, so without loss of generality assume that 0 is such a point and that $\partial\tilde{\varphi}^{-1}(0)$ is bounded. If F is a fundamental domain of Γ_X containing i, then the geodesic $[i, 0)$ passes through infinitely many translates of the fundamental domain $\{\gamma_n F\}_{n=1}^{\infty}$ where $\gamma_n \in \Gamma_X$. Let $\gamma_n(i) \in \gamma_n F$ be translates of i and iy_n be orthogonal projection of $\gamma_n(i)$ onto the imaginary line. It follows that both $\gamma_n(i)$ and y_n converge to 0 in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ as they lie in the same translate of F. The maps $g_n(z) = y_n z$ and $k_n(z) = \gamma_n^{-1} g_n(z)$ are isometries of $\mathbb{H}^2$ and $h_n = g_n^{-1}\left(\partial\tilde{\varphi}^{-1}\right) g_n$ is a homeomorphism of $\partial\mathbb{H}^2$. We claim the following:

1. *The sequence (h_n) converges point-wise to the map $\xi \to a\xi$ on the boundary where a is the derivative of $\partial\tilde{\varphi}^{-1}$ at* 0: composing by an isometry of $\mathbb{H}^2$, it can be assumed without loss of generality that $\partial\tilde{\varphi}^{-1}(0) = 0$ and $\partial\tilde{\varphi}^{-1}(\infty) = \infty$. Since $\partial\tilde{\varphi}^{-1}$ is differentiable at 0 it follows that

$$h_n(\xi) = a\xi + r(y_n\xi)\xi$$

   where $r(\xi) \to 0$ as $\xi \to 0$ and thus as $n \to \infty$ we have $h_n(\xi) \to a\xi$.

2. *The sequence k_n has a convergent subsequence which converges to some $k \in PSL(2, \mathbb{R})$*: Using the fact that $k_n(i) \to i$ as $n \to \infty$ it is clear that k_n lies in some compact set of $\mathrm{PSL}(2, \mathbb{R})$, and thus there is a subsequence of k_n which

converges to k in the $\mathrm{PSL}(2,\mathbb{R})$ topology. This means that $k_{n_j} \to k$ pointwise as well.

Now let $G = \mathrm{Homeo}^+(\partial\mathbb{H}^2)/\mathrm{PSL}(2,\mathbb{R})$ where the quotient is by right action. Then using equivariance and the two claims above the identity in G can be written as:

$$\begin{aligned}
\left[\mathrm{id}_{\mathbb{H}}^2\right] &= [\{z \mapsto az\}] = \left[\lim_{n\to\infty} h_n\right] \\
&= \left[\lim_{n\to\infty} g_n^{-1}\partial\tilde{\varphi}^{-1}g_n\right] = \left[\lim_{n\to\infty} \partial\tilde{\varphi}^{-1}g_n\right] \\
&= \left[\lim_{n\to\infty} \partial\tilde{\varphi}^{-1}\gamma_n k_n\right] = \left[\lim_{n\to\infty} \partial\tilde{\varphi}^{-1}k_n\right] \\
&= \left[\partial\tilde{\varphi}^{-1}k\right]
\end{aligned}$$

This shows that $\partial\tilde{\varphi}$ agrees with an isometry g on the boundary. Using Alexander's trick it follows that $\bar{\varphi}$ is isotopic to the isometry. Since $\bar{\varphi}$ is equivariant it follows that g is equivariant when restricted to the boundary and thus everywhere. The equivariance implies that g induces an isometry from $X \to Y$ which is isotopic to φ. This contradicts the fact that $\mathfrak{X}$ and $\mathfrak{Y}$ are distinct elements of $\mathrm{Teich}(S)$. $\square$

Proof of theorem II.9. Assuming $\mathfrak{X}$ and $\mathfrak{Y}$ be distinct elements of $\mathrm{Teich}(S)$ we will first show that $K(\mathfrak{X},\mathfrak{Y}) > 0$. The idea is to construct a closed curve c whose length in Y is larger than in X using the fact that the function $D(z)$ is unbounded. For this we construct a c so that it is a geodesic in Y and estimate a lower bound on it's length in Y. Then pull back c to X and estimate an upper bound on it's length in X.

Start with points 0 and z in $\mathbb{H}^2$, and let $\varphi : X \to Y$ be the homeomorphism gf^{-1}. Let y_0, y_1 be the projections of $w_0 = \tilde{\varphi}(0), w_1 = \tilde{\varphi}(z)$ under p_Y. Let $\gamma \in \Gamma_Y$ be such that γw_0 is the translate of w_0 closest to w_1. The piecewise geodesic $\tilde{c}_0$ from w_0 to w_1 and then w_1 to γw_0 projects down in Y to a closed curve c_0 in Y based at y_0 and passes through y_1. Let $(y_0, \dot{c}_0(0))$ be an element of T^1Y and N_ϵ be an ϵ neighborhood of $(y_0, \dot{c}_0(0))$. Since geodesic flow Φ_t on T^1Y is area preserving, by Poincare Recurrence theorem the set of recurrence points are dense in N_ϵ which means that there is a $(y_2, v_2) \in \mathrm{T}^1Y$ such that after a finite time $T > 0$ the geodesic orbit of the flow returns back to N_ϵ, i.e. $\Phi_T(y_2, v_2) \in N_\epsilon$. The orbit $\Phi_t(y_2, v_2)$ where $0 \le t \le T$ can be approximated by closed geodesic orbit based at some $(y_3, v_3) \in N_\epsilon$ using Anosov's closing lemma (see [AK95]). Let the projection of this geodesic in Y be the closed geodesic c based at y_3. Note that the length of c is larger than the length of the geodesic interval $[y_0, y_1] \subset c_0$ as $(y_0, \dot{c}_0(0))$ and (y_3, v_3) are in an $\epsilon-$neighborhood and thus they fellow travel till y_1. Consider the lift $\tilde{c}$ of c in $\mathbb{H}^2$ based at the point w_2

where w_2 is chosen to be in the same translate of the fundamental domain as w_0.

$$\ell_Y(c) \geq d(w_0, w_1) = d(\tilde{\varphi}(0), \tilde{\varphi}(z)) \geq d(0, z) + D(z)$$

The lift of the pullback $\varphi^{-1}(c)$ based at $z_1 = \tilde{\varphi}^{-1}(w_2)$ has an endpoint at $\gamma' z_1 = \varphi_*^{-1}(\gamma)\tilde{\varphi}^{-1}(w_2)$ because of equivariance, where $\gamma' \in \Gamma_X$.

$$\begin{aligned}
\ell_X(c) &\leq d(z_1, \gamma' z_1) \\
&\leq d(z_1, 0) + d(0, z) + d(z, \gamma' z_1) \\
&\leq \lambda d(\tilde{\varphi}(z_1), \tilde{\varphi}(0)) + \lambda\epsilon' + d(0, z) + \lambda d(\tilde{\varphi}(z), \tilde{\varphi}(\gamma' z_1)) + \lambda\epsilon' \\
&= \lambda d(w_2, w_0) + 2\lambda\epsilon' + d(0, z) + \lambda d(w_1, \gamma w_2) \\
&\leq \lambda\epsilon + 2\lambda\epsilon' + d(0, z) + \lambda T
\end{aligned}$$

where (λ, ϵ') are the quasi-isometric constants of $\tilde{\varphi}$. This means that the ratio:

$$\frac{\ell_Y(c)}{\ell_X(c)} \geq \frac{d(0, z) + D(z)}{d(0, z) + \lambda T + \lambda\epsilon + 2\lambda\epsilon'}$$

Since Y is compact the time of return T has an upper bound, on the other hand due to lemma II.10 one can always choose a z so that $D(z)$ is arbitrarily large. Thus $K(\mathfrak{X}, \mathfrak{Y}) > 0$.

Using the fact that $K(\mathfrak{X}, \mathfrak{Y}) \leq L(\mathfrak{X}, \mathfrak{Y})$ it follows that K is 0 if and only if $\mathfrak{X} = \mathfrak{Y}$. □

This completes the proof of that L and K are metrics on Teich(S). Some more properties of K will be proved later on in this chapter.

II.3. Thurston Stretch Maps

In this section the Thurston's Stretch maps are constructed and it is shown that they are geodesics in the Lipschitz metric L.

Let Δ be an ideal hyperbolic triangle. Let $\mathscr{F}$ be the (partial) foliation on Δ where each leaf is a horocycle from one of the vertices of Δ and let $\mathscr{G}$ be the (partial) foliation on Δ where each leaf is a geodesic emerging from one of the vertices of the triangle. Let C be the unfoliated region of Δ. See figure II.2. Consider the map $\varphi_t : \Delta \to \Delta$ which maps a point $z \in \Delta - C$ which is a distance r away from the central region to the point $\varphi_t(z)$ along $\mathscr{G}$ which is a distance $e^t r$ away from C, and φ_t is constant on C.

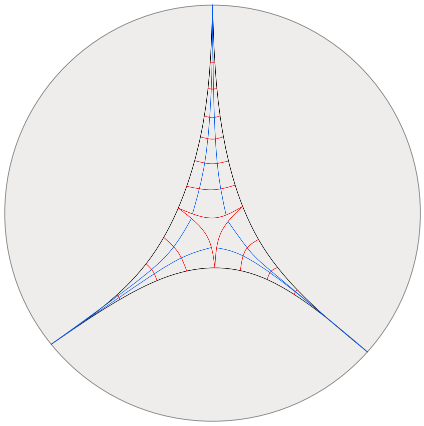

Figure II.2: The red curves are leaves of $\mathscr{F}$ and the blue curves are leaves of $\mathscr{G}$.

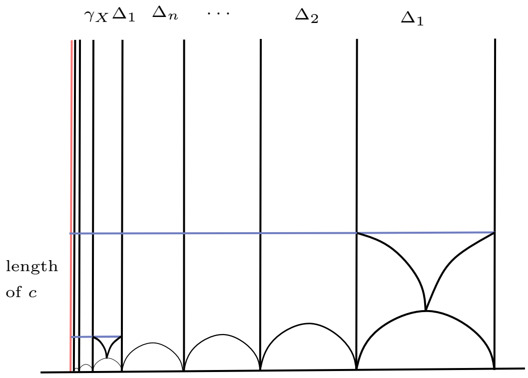

Figure II.3: The red geodesic is c, the vertical geodesics are lifts of γ_i.

Proposition II.11. *The map φ_t is an e^t-Lipschitz homeomorphism on Δ and on the boundaries of Δ are φ_t achieves the Lipschitz constant e^t.*

Proof. Let $K = e^t$ and assume that the vertices of the triangle lie at $0, 1$, and ∞. Since all components of $\Delta - C$ are isometric to each other, consider any component P of $\Delta - C$ and using the isometries of $\mathbb{H}^2$ assume that the vertex of Δ in P is at ∞ (in the plane model). Explicitly the map φ_t on P can be written as $\varphi_t\big|_P(x+iy) = x+iy^K$. Since,

$$\frac{\| \left(\mathrm{d}\varphi_t\right)_z(\xi)\|^2}{\|\xi\|^2} \leq \frac{\xi_x^2 + K^2\xi_y^2}{\|\xi\|} \leq K^2$$

it follows that $\| \left(\mathrm{d}\varphi_t\right)_z \|_{\mathrm{op}}^2 \leq K$, which means that φ_t is K-Lipschitz on P as the edges are exactly stretched by K. Since the norm is an isometry invariant it follows that $\| \left(\mathrm{d}\varphi_t\right)_z \|_{\mathrm{op}}^2 \leq K$ for all $z \in \Delta - C$. $\square$

Definition II.12. Let Δ_1 and Δ_2 be two ideal hyperbolic triangles in $\mathbb{H}^2$ which share a common geodesic γ. Orient γ so that Δ_1 is to the left of Δ_2. Let z_1 and z_2 be the points on γ where the central regions of Δ_1 and Δ_2 meet γ. The *shear* of Δ_1 with respect to Δ_2 along γ, denoted $\mathrm{shear}(\Delta_1, \Delta_2)$, is defined as the signed distance between z_1 and z_2 along the orientation of γ.

Proposition II.13. *Let X be a compact hyperbolic surface with geodesic boundary and let c be a boundary component. If λ is a maximal lamination on X and $\{\gamma_i\}_{i=1}^n \in \lambda$ are leaves spiraling onto c then the length of c is determined by a linear combination of the n shear parameters corresponding to the leaves γ_i.*

Proof. The triangles $\Delta_1, \cdots \Delta_n$ in X are such that Δ_i and $\Delta_i + 1$ share the geodesic γ_{i+1} where the index is modulo n. In the lift, we get a sequence of triangles (which are translates of Δ_i) converging onto the lift of the geodesic c. Consider the case when $\text{shear}(\Delta_i, \Delta_{i+1}) > 0$ for all i modulo n. The picture in figure II.3 shows the lift of the triangles with edges γ_i to the universal cover embedded into $\mathbb{H}^2$ using some developing map. The length of c is the same as the distance between the extreme horocycles of consecutive translates of any Δ_i (as shown in figure II.3). It follows that length of c will be $\sum_{i=1}^{n} \text{shear}(\Delta_i, \Delta_{i+1})$ where Δ_{n+1} is a translate of Δ_1. All other cases can be argued similarly. $\square$

Given a closed hyperbolic surface X and a maximal lamination λ on X with finitely many leaves, the completion of $X - \lambda$ consists of finitely many ideal hyperbolic triangles $\Delta_1, \cdots, \Delta_n$. The edges of these triangles correspond to the leaves of λ. Let $\text{shear}(\Delta_i, \Delta_j)$ be the shear of Δ_i with respect to Δ_j whenever both Δ_i and Δ_j have an edge which corresponds to the same leaf of λ in X. Using the above proposition each of these triangles can be stretched by an e^t Lipschitz map for a fixed $t > 0$. Gluing back the stretched triangles in right order with shear coordinates $e^t\text{shear}(\Delta_i, \Delta_j)$, whenever Δ_i and Δ_j are glued, gives a hyperbolic surface X_t and an e^t-Lipschitz map from $\varphi_t : X \to X_t$. If $f : S \to X$ was a marking in Teich(S) then the above construction gives a new point in $\varphi_t f : S \to X_t$ in Teich(S). This gives a ray in Teich(S) based at $\mathfrak{X}$ which we will represent by $\mathfrak{X}_t^\lambda$. Since $\varphi_t : X \to X_t$ is e^t-Lipschitz it follows that $L(\mathfrak{X}, \mathfrak{X}_t^\lambda) \leq t$. From theorem I.27 it follows that any leaf of λ which corresponds to the edge of some Δ_i spirals onto a closed geodesic c in X. The ratio $\ell_{\mathfrak{X}_t^\lambda}(c)/\ell_{\mathfrak{X}}(c)$ is exactly e^t as by the above proposition $\ell_{\mathfrak{X}}(c)$ is determined by a linear combination of all the shear parameters corresponding to leaves of λ which spiral onto c and by construction these shear parameters are scaled by e^t. This means that $t \leq K(\mathfrak{X}, \mathfrak{X}_t^\lambda) \leq L(\mathfrak{X}, \mathfrak{X}_t^\lambda) \leq t$. This means that the curve $\mathfrak{X}_t^\lambda$ is a geodesic in Thurston's metric. This ray is called the *Thurston Stretch ray* and the map $\varphi_t : X \to X_t$ is called a stretch map. This construction has more subtleties when the lamination is allowed to have infinitely many leaves. For the general construction see [Thu98]. We summarize the general case of the discussion above in the following theorem:

Theorem II.14 (Stretch Rays are Geodesics)**.** *Let $\mathfrak{X} \in$ Teich(S) and λ be a maximal geodesic lamination on X. Then the stretch ray $\mathfrak{X}_t^\lambda$ based at $\mathfrak{X}$ is a geodesic in both the metrics L and K.*

For a better understanding of stretch maps it is good to look at them explicitly for some hyperbolic surface.

Example II.15 (Stretch Maps on Pants)**.** Let X be a hyperbolic pair of pants with geodesic boundary of lengths ℓ_1, ℓ_2, ℓ_3. By a simple area argument, any maximal lamination λ on X cuts it into 2 triangles Δ_1, Δ_2. The goal of this example is to observe how does a horocycle h on X change under the stretch map. This is what I mean: in the universal cover the triangles Δ_1 and Δ_2 lift to triangles and let $\tilde{h}$ be a segment of a horocycle, as in figure II.4, starting from an edge of Δ_1 and ending at the lift of the boundary component of length ℓ_1. In this particular case, $\tilde{h}$ is a horizontal segment $\Im(z) = 1$ and without loss of generality assume that it starts from $z = i$, see figure II.4. Let h be the corresponding curve in X. We want to understand two things: (a) how does the length of h change under φ_t and (b) how does the image of h "wrap around" the boundary in X? Denote the shear of the two triangles with respect to each edge as α, β, γ. First we calculate the real part of vertices of the triangles with vertical edges. Let $u_0 = 0$ and $v_0 = -1$ and u_n, v_n be alternating labels for the vertices, then

$$u_n = -v_{n-1} - e^{-n\alpha-(n-1)\beta}, \; n \geq 1$$
$$v_n = -u_n - e^{-n\alpha-n\beta}, \; n \geq 0$$

The sequence $(u_0, v_0, u_1, v_1, \cdots)$ converges to the endpoint x_0 of the lift of the boundary of length ℓ_1 which corresponds to the series:

$$x_0 = -\sum_{n\geq 0} e^{-n\alpha-n\beta} - \sum_{n\geq 1} e^{-n\alpha-(n-1)\beta} = -\frac{1+e^{-\alpha}}{1-e^{-\ell_1}}$$

where $\ell_1 = \alpha + \beta$ is used. The length of h is

$$\ell(h) = \frac{1+e^{-\alpha}}{1-e^{-\ell_1}}$$

Since this is a decreasing function and the half leaf of the horocycle h is again mapped to a half leaf of a horocycle, it follows that under the stretch map the length of h decreases as both ℓ_1 and α are stretched by e^t. This answers the first question.

To understand the second question we need to figure out a way to quantify what it means to "wrap around" a boundary component on Pants. Let C be the unique perpendicular between the boundary of length ℓ_1 and any one of the other boundaries. The preimage $p_X^{-1}(C)$ is a collection of circular arcs which are perpendicular to the line $\Re(z) = x_0$. Each component of $p_X^{-1}(C)$ intersects $\tilde{h}$ atmost once and moreover only finitely many components intersect $\tilde{h}$. I will say that h wraps around the boundary $n-$times if it intersects $p_X^{-1}(C)$ at $n-$points. Let $\tilde{C}_0$ be the particular lift of C such that any other lift with a smaller radius does not intersect $\tilde{h}$. Such a $\tilde{C}_0$ can be chosen as only finitely many of the lifts intersect $\tilde{h}$. Explicitly the components of $p_X^{-1}(C)$ can be indexed by $n \in \mathbb{Z}$ so that the equation of $\tilde{C}_n$ is

$$(x - x_0)^2 + y^2 = R^2 e^{2n\ell_1}$$

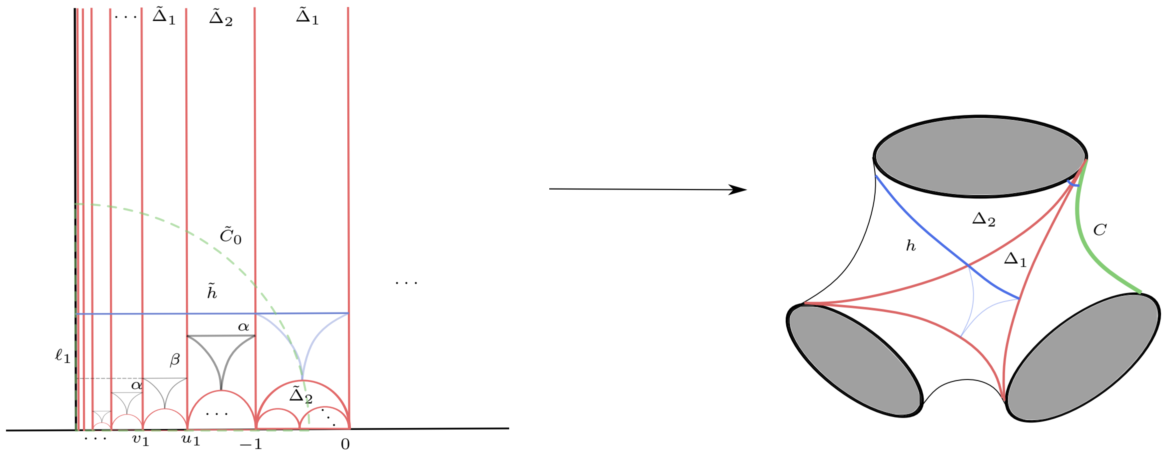

Figure II.4: Triangulation of pants.

where R is the radius of $\tilde{C}_0$. By construction $\tilde{C}_n$ intersects $\tilde{h}$ only for $n \geq 0$. If $\tilde{C}_n$ intersects $\tilde{h}$ then it follows that

$$(x - x_0)^2 = R^2 e^{2n\ell_1} - 1$$

The first lift $\tilde{C}_N$ with $N > 0$ which does not intersect $\tilde{h}$ is such that the intersection point of $\tilde{C}_N$ with $\Im(z) = 1$ has strictly positive real part. Thus,

$$x_0^2 < R^2 e^{2N\ell_1} - 1 \implies N > \frac{1}{2\ell_1} \log\left(\frac{x_0^2 + 1}{R^2}\right) \tag{II.7}$$

This means that $\tilde{C}_n$ intersects $\tilde{h}$ if and only if

$$0 \leq n \leq \left\lceil \frac{1}{2\ell_1} \log\left(\frac{x_0^2 + 1}{R^2}\right) \right\rceil$$

Note that the right hand side is a decreasing function of the shear parameters α and β, implying that the number of intersection points of $p_X^{-1}(C)$ and $\tilde{h}$ decreases as the shear increases. Thus the stretch map φ_t "unwraps" h around the boundary of length ℓ_1.

Remark. Another interesting thing to take note of is that the stretch map is affine on the boundary of any hyperbolic surface X. This follows from the fact that in the vertical lift of the boundary the stretch map restricts to the map $x_0 + iy \mapsto x_0' + iy^K$ where $K = e^t$ (see figure II.6). This corresponds to the affine map $x \mapsto Kx$ on the affine universal cover of the boundary component in X.

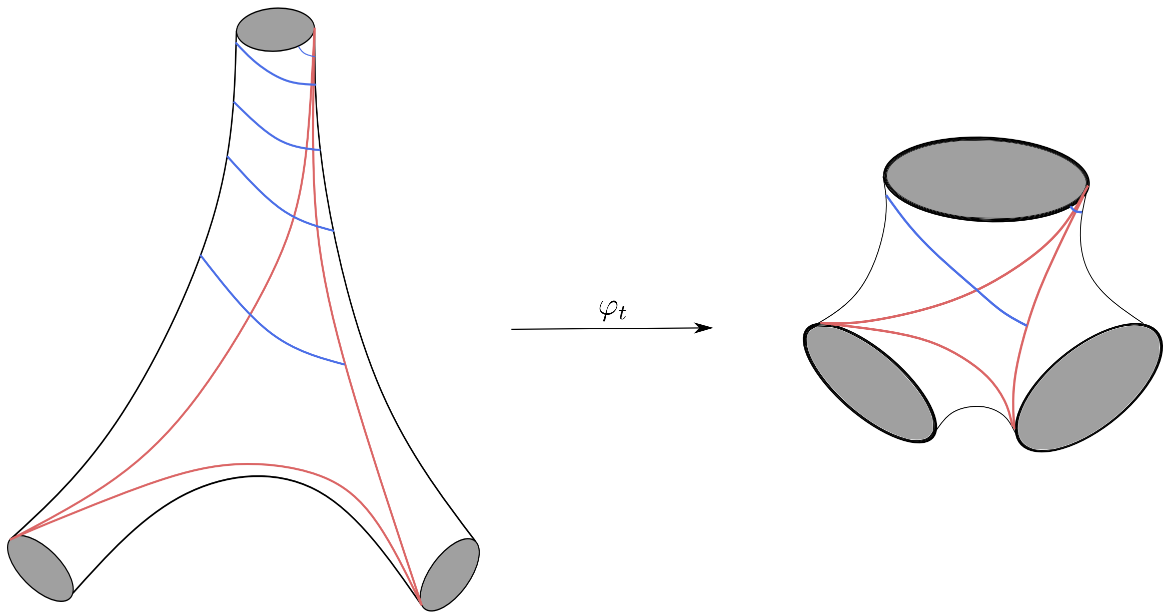

Figure II.5: This picture summarizes the entire discussion above. The stretch maps expands the boundary by e^t, decreases the length of h while unwrapping it around the boundary.

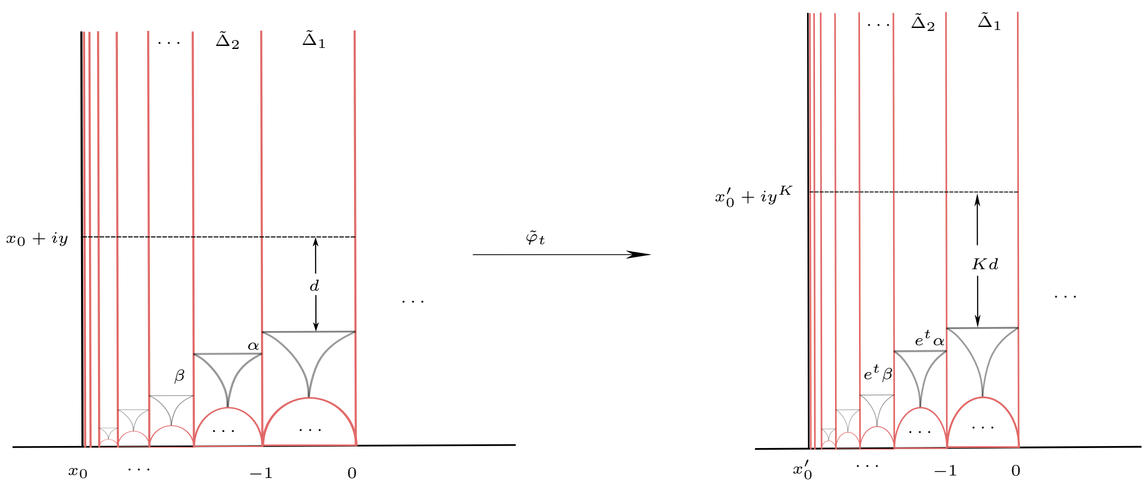

Figure II.6: The stretch map restricts to an affine map on the boundary.

Chapter III

Orthogeodesic Foliation, Arc Complex, and Dilation Rays

In this chapter I will give the construction of *Dilation Rays* in the Teichmüller space of a closed surface S given a geodesic lamination λ, as done in [CF21]. Our main focus will be on the special case when λ only contains simple closed geodesics which, by theorem I.27, can only have finitely many such leaves. Such geodesic laminations are called *multicurves*. In the case when λ is a pants decomposition of S it is shown that the corresponding dilation rays are Thurston geodesics.

Given a closed surface S of genus $g \geq 2$ let $f : S \to X$ be a hyperbolic marking. Let λ be a geodesic lamination on X consisting of only simple closed geodesics and Y be the completion of a component of $X - \lambda$. For arbitrary geodesic laminations Y will be a crowned surface, but in our specific case Y is just a surface with boundary. Let Σ be the underlying topological surface corresponding to Y. The construction of the following objects for arbitrary laminations is almost exactly the same as what is presented here and can also be found in [CF21].

III.1. Orthogeodesic Foliation

Definition III.1. The *valency* of a point $y \in Y$ is the cardinality of the set $\{p \in \partial Y \mid d(y,p) = d(y,\partial Y)\}$. The *spine* of a hyperbolic surface is defined as $\mathrm{Sp}(Y) = \{y \in Y \mid \text{valency of } y \geq 2\}$. Further let $\mathrm{Sp}_k(Y)$ be the set of all points in $\mathrm{Sp}(Y)$ with valency exactly k.

The spine can be thought of as a graph with vertices being points $y \in \mathrm{Sp}(Y)$ with valency strictly greater than 2 and geodesic edges e being the arcs connecting vertices where each point is of valency exactly 2. The generic vertex of the spine is of valency 3 as given three geodesics in $\mathbb{H}^2$ such that the half planes bounded by them are disjoint then there is always a point equidistant from all three geodesics. Points of valency larger than 3 occur when there is more symmetry in the surface.

Example III.2. For example, consider the pair of pants of boundary lengths (ℓ_1, ℓ_2, ℓ_3). There are three cases of the pants:

1. The lengths satisfy the strict triangle inequality: $\ell_i + \ell_j < \ell_k$ where (ijk) is a permutation of (123).

2. One of the lengths is strictly larger than sum of the other two: $\ell_1 > \ell_2 + \ell_3$.

3. One of the lengths is the sum of the other two: $\ell_1 = \ell_2 + \ell_3$.

In the first two cases the spine consists of only two vertices of valency 3 and edges, while in the last case there is single vertex of valency 4. These are explicitly shown in figure III.1.

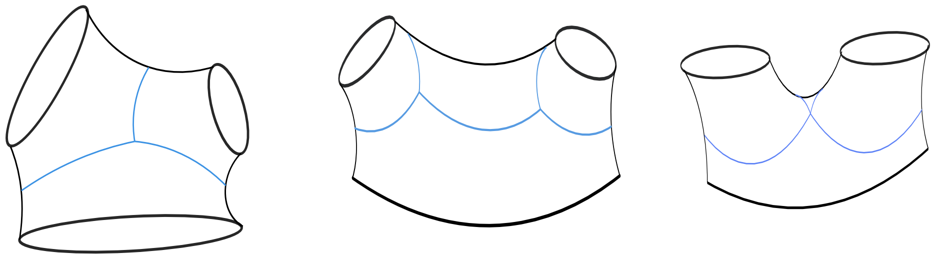

Figure III.1: This shows the spine on pants in the three cases discussed above.

Proposition III.3. *The surface Y is homotopically equivalent to $Sp(Y)$.*

Proof. Consider the map $r : Y \to \mathrm{Sp}(Y)$ which maps $y \in Y - Sp(Y)$ to the point of intersection $r(y) \in Sp(Y)$ of the extension of the perpendicular geodesic α_y from y to ∂Y realizing the distance $d(y, \partial Y)$ and the spine; and r is identity on the spine. The fibers of r are geodesics perpendicular to ∂Y at both ends. For any $y, y' \in Y - \mathrm{Sp}(Y)$ the geodesics α_y and $\alpha_{y'}$ are either disjoint or one is contained in the other. This map is continuous as any open interval I of an edge of the spine the set $r^{-1}(I)$ is an open strip on the surface.

For any $y \in Y$ let $\gamma_y(t)$ be a unit speed parametrization of a sub-arc of $r^{-1}(r(y))$ such that $\gamma_y(1) = r(y)$ and $\gamma_y(0) = y$. Define the homotopy $H(y,t) = \gamma_y(t)$. Then $H(y,t) = \gamma_y(t)$ is a homotopy with $H(y,1) = r(y)$, $H(y,0) = y$, and $H\big|_{\mathrm{Sp}(Y)}(y,t) = y$. Thus H is a deformation retract from the identity map to $i \circ r$, where $i : \mathrm{Sp}(Y) \to Y$ is the natural inclusion. Thus Y is homotopically equivalent to it's spine. $\square$

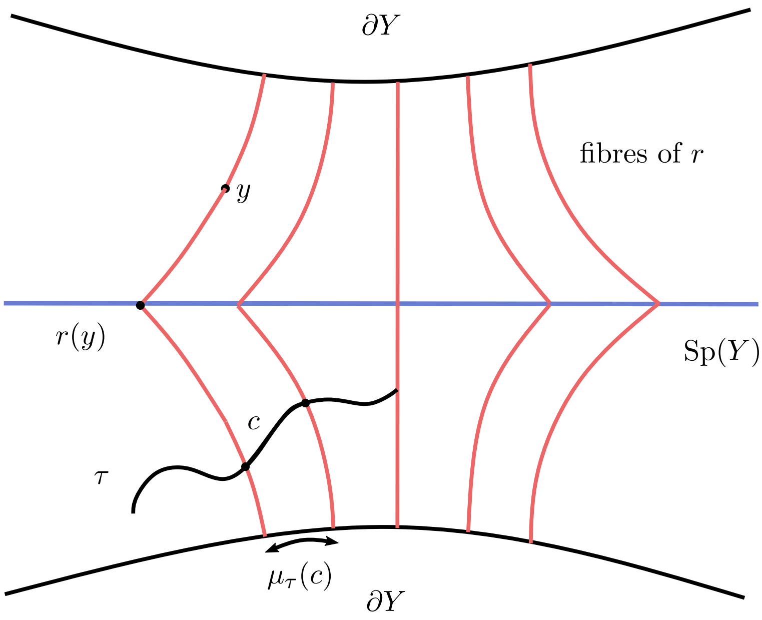

Figure III.2: Orthogeodesic foliation near an edge of the spine.

Given any two points $y, y' \in \mathrm{Sp}_2(Y)$ which lie on the same edge e of the spine the arcs $r^{-1}(y)$ and $r^{-1}(y')$ are homotopic to each other relative to the boundary ∂Y. A dual arc α_e is the homotopy class relative to ∂Y of fibers of r corresponding to e. There is a special representative of the class α_e which is the fiber perpendicular to e as well as the boundary. This representative will also be denoted by α_e. Denote by $\underline{\alpha} = \bigcup_{e \subset \mathrm{Sp}(Y)} \alpha_e$, this is called the dual arc system.

If τ is an arc in Y transverse to the fibers of r such that $r(\tau) \subset e$ for some edge e of $\mathrm{Sp}(Y)$ then define a measure μ_τ on τ by defining the measure of any sub-arc c to be length of the curve on ∂Y obtained by continuously deforming c to the boundary keeping each point of c on the same fiber of r. For an arbitrary transverse arc τ, it can be decomposed into finitely many components τ_i such that $r(\tau_i) \subset e_i$ for some edge e_i and define $\mu_\tau = \sum_i \mu_{\tau_i}$. See figure III.2 for a cartoon of this construction.

Proposition III.4. *μ defines a transverse measure on the fibers of r.*

Proof. The measure is well defined as no matter what homotopy is used the measure is going to be the length of the arc between the end-points of the corresponding fibers at the boundary. It is clear that given arcs $\tau' \subset \tau$ it follows that $\mu_\tau\big|_{\tau'} = \mu_{\tau'}$ as the length at boundary is going to be the same for both.

Given two arcs τ and τ' which are homotopic relative to the fibers of r (i.e. each point remains on the same fiber of r) the measures of μ_τ and $\mu_{\tau'}$ are the same as being "homotopic relative to fibers of r" is an equivalence relation. $\square$

Definition III.5 (Orthogeodesic Foliation on Y)**.** The measured foliation $\mathcal{O}(Y, \partial Y)$ on Y where the leaves are fibers of r and the transverse measure is defined as above by μ. This foliation has singularities of order $k-2$ at the vertices of $\mathrm{Sp}(Y)$ of valency k.

Remark. The measured foliation defined above is not a smooth foliation as at the spine the leaves are not smooth. But we only care about measured foliations upto homotopy and there is a smooth measured foliation in the same homotopy class of $\mathcal{O}(Y, \partial Y)$.

Definition III.6 (Orthogeodesic foliation on X)**.** Given a closed surface X with a multicurve λ, the foliation $\mathcal{O}_\lambda(X)$ obtained by constructing the orthogeodesic foliations $\mathcal{O}(Y, \partial Y)$ for each component Y in $X - \lambda$ and then gluing them back together without any twisting.

Calderon and Farre also show that $\mathcal{O}_\lambda : \mathrm{Teich}(S) \to \mathcal{MF}(S)$ is a homeomorphism onto it's image (see [CF21]).

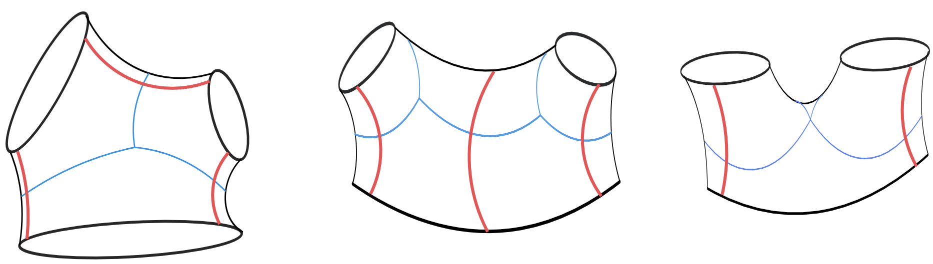

Figure III.3: This figure shows the dual arc system for the three cases of hyperbolic pants discussed earlier.

III.2. Arc Complex and Filling Arcs

Let S, X, Y, Σ be as in the previous section. An arc α in Σ with end points in $\partial\Sigma$ is said to *essential* if it cannot be homotopically deformed to the boundary.

Definition III.7. An *arc system* $\underline{\alpha} = \bigcup_i \alpha_i$ on Σ is a union of homotopy classes of simple essential arcs such that there is a choice of representative arcs a_i for the classes α_i in the arc system such that $a_i \cap a_j = \emptyset$. Often we abuse notation and denote just the arcs a by α as well.

Definition III.8. A *weighted arc system* $\underline{A}$ on Σ is formally denoted by $\sum_i c_i\alpha_i$ where $\{\alpha_i\}$ forms an arc system and $c_i \in \mathbb{R}^+$.

The dual arc system $\underline{\alpha}(Y)$ is an example of an arc system. An example for a weighted arc system is $\underline{A}(Y) = \sum_{e\subset \mathrm{Sp}(Y)} \mu_e(e)\alpha_e$. Note that if Y and Y' represent the same point in Teich(Σ) then the isometry preserves the weighted dual arc system. This means that $\underline{A}(\Box)$ can be defined as a function on Teich(S). Now we describe the co-domain and range of this function. An arc system $\underline{\alpha}$ on Y is said to be *filling* if every component of $Y - \underline{\alpha}$ is a topological disk.

Proposition III.9. *The dual arc system is a filling arc system.*

Proof. It is enough to show that each component C of $Y - \underline{\alpha}(Y)$ is contractible. Using restriction of the map r to C it can be deformation retracted onto the component of the spine in C. Since in each component C there can be exactly one vertex of the spine, the set $\mathrm{Sp}(Y)\cap C$ is a star shaped graph and thus can be contracted to a single point. This proves that C is contractible. $\Box$

Definition III.10 (Arc Complex of Σ)**.** The arc complex $\mathscr{A}(\Sigma, \partial\Sigma)$ of Σ is a simplicial complex defined as follows:

1. The $0-$simplexes are homotopy classes of simple essential arcs.

2. The vertices $(\alpha_1, \cdots, \alpha_n)$ span an $n-$simplex if $\underline{\alpha} = \bigcup_{i=1}^n \alpha_i$ is an arc system.

The sub-complex $\mathscr{A}_\infty(\Sigma, \partial\Sigma)$ of $\mathscr{A}(\Sigma, \partial\Sigma)$ only has simplexes whose vertices form a non-filling arc systems. The compliment of the non-filling arc complex is called the filling arc space and denoted $\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)$. The geometric realization space of $\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)$, denoted $|\mathscr{A}_{\text{fill}}(\Sigma, \partial\Sigma)| \times \mathbb{R}^+$, is the space of all weighted filling arc systems.

Example III.11. Again as an example we look at the pair of pants to construct an example of the arc complex. By proposition 2.2 in [FM11] we know that any two arcs connecting the same boundary components are homotopic to each other. Thus the arc complexes have exactly 6 vertices: one homotopy class of arcs for each pair of boundary components and one homotopy class of arcs for a single boundary component. The arc complex $\mathscr{A}(S_{0,3}, \partial S_{0,3})$ is shown in figure III.4.

Theorem III.12 ([Luo07, CF21])**.** *The map* $\underline{A}$: *Teich*$(\Sigma) \to |\mathscr{A}_{\mathit{fill}}(\Sigma, \partial\Sigma)| \times \mathbb{R}^+$ *which maps* $\mathfrak{Y}$ *to the dual arc system* $\underline{A}(Y)$ *is a homeomorphism.*

Remark. Corollary 1.4 in Luo's paper proves this result for "ideal triangulation" of surfaces with boundary which cuts the surfaces into right angled hexagons. This is

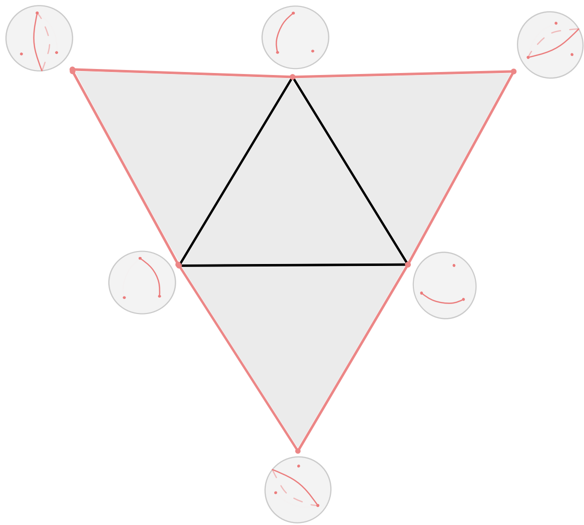

Figure III.4: Arc complex of the topological pants $S_{0,3}$. The red colored boundary of the complex in this is the non-filling sub-complex.

the generic case for the dual arc system as in the generic case the weights $\mu_e(e)$ match with the radius coordinate $z(e)$ defined by Luo. Calderon and Farre proved this result as stated here in greater generality for crowned surfaces.

III.3. Dilation Rays

Definition III.13 (Dilation Rays)**.** Let $\mathfrak{X} \in \text{Teich}(S)$ and λ be a multicurve in X. Consider the completion $Y_1, \cdots, Y_n$ of the components of $X - \lambda$ and the corresponding points $\mathfrak{Y}_i \in \text{Teich}(\Sigma_i)$. Then $\underline{A}^{-1}\left(e^t \underline{A}(\mathfrak{Y}_i)\right)$ defines a curve in $\text{Teich}(\Sigma_i)$ denoted by $\mathfrak{Y}_i(t)$. Then gluing all $Y_i(t)$ together without any twisting gives a curve $\mathfrak{X}_t^\lambda$ in $\text{Teich}(S)$. This curve is called the *Dilation ray* based at X.

Remark. The marking on Y_i is determined by the marking on X by restrictions. The marking on X_t^λ is determined by the markings on each $Y_i(t)$. If $f_i : \Sigma_i \to Y_i(t)$ are markings then there is a natural choice for $f : S \to X_t^\lambda$ which comes from the universal property of the pushout in the category of topological spaces. By construction of the inverse map $\underline{A}^{-1}$, markings on $\mathfrak{Y}_i$ are such that the arc α_e in Y_i maps to the special representative of the dual arc in $Y_i(t)$.

A natural question to ask is whether dilation rays are geodesics? This question is partially answered in this section.

Lemma III.14. *If for each $t > 0$ there exists e^t-Lipschitz homotopy equivalence $Y_i \to Y_i(t)$ which homeomorphically maps boundary to boundary and preserves the special representatives of the dual arc system upto homotopy relative to the endpoints* *and expands each boundary component of Y_i by a factor of e^t for all i, then the dilation ray $\mathfrak{X}_t^\lambda$ is a Thurston geodesic.*

Proof. Let Y_i have boundary lengths $(\ell_1, \cdots, \ell_n)$ in the decomposition $X - \lambda$ and then $\mathfrak{Y}_i(t) = \underline{A}^{-1}(e^t \underline{A}(\mathfrak{Y}_i))$ is the surface with boundary lengths $e^t \cdot (\ell_1, \cdots, \ell_n)$ because of theorem 1.2 in [Luo07]. By assumption there is an e^t-Lipschitz map $\varphi_i^t : Y_i \to Y_i(t)$ which preserves the special representatives of the dual arcs upto homotopy relative to endpoints and stretches the boundary by a factor of e^t. Since the endpoints of the dual arcs are fixed on the boundary of each Y_i by φ_i, gluing all $(Y_i(t), \varphi_i^t f_i)$ without any twisting results in a marking $(X_t^\lambda, \varphi_t f)$, where φ_t is the pushout of all φ_i^t, and $[(X_t^\lambda, \varphi_t f)]$ is exactly the dilation ray $\mathfrak{X}_t^\lambda$. Let c be any closed curve in X, and c' be the arc of c which lies in Y. Since φ_t is e^t Lipschitz the length of c' stretches by less than e^t. This means that after gluing the length of c also increases by a factor which is less than e^t. But since the multicurve λ stretches exactly by e^t we must have $K(\mathfrak{X}, \mathfrak{X}_t^\lambda) = t$. □

From here I will call maps which satisfy assumption of III.14 *dilation maps.* Now if we can construct a dilation maps for any hyperbolic surface Y with geodesic boundary we can conclude that dilation rays are indeed Thurston geodesics. But this construction is not very easy. In the remainder of this section the construction of such dilation maps for hyperbolic pair of pants is given in order to prove the following result.

Theorem III.15. *If $\mathfrak{X} \in Teich(S)$ and λ is a pants decomposition of S, then the dilation ray $\mathfrak{X}_t^\lambda$ is a Thurston geodesic.*

Here is a rough outline of the construction. Due to Thurston we already have stretch maps which stretch the boundary of pants by a factor of e^t, but do not preserve the dual arc representatives. The key idea in the construction is to take the "average" of two stretch maps defined over different triangulations with opposite orientations and show that this average map preserves the dual arc representatives. First we define the average of Lipschitz map as done in [GK17].

Proposition III.16 (Barycenter in $\mathbb{H}^2$, Lemma 2.11 in [GK17])**.** *Let $I = \{1, \cdots, k\}$ for $k \geq 1$ and $\underline{r} = \{r_i \in \mathbb{R}^+\}_{i \in I}$ are such that $\sum_i r_i = 1$. Let*

$$\left(\mathbb{H}^2\right)^I_{\underline{r}} = \{(p_i)_{i\in I} \mid \sum_{i \in I} r_i d(p_1, p_i)^2 < \infty\}$$

Then there is a map

$$\mathbf{m}_{\underline{r}} : \left(\mathbb{H}^2\right)^I_{\underline{r}} \to \mathbb{H}^2$$

which maps $(p_1, \cdots, p_n)$ to the minimizer of $\sum_i r_i d(\cdot, p_i)^2$. Moreover, $\mathbf{m}_{\underline{r}}$ has the following properties:

1. $\mathbf{m}_{\underline{r}}$ *is* r_i-*Lipschitz in the* $i-$*th entry, i.e.*

$$d(\mathbf{m}_{\underline{r}}(p_1, \cdots, p_k), \mathbf{m}_{\underline{r}}(q_1, \cdots, q_n)) \leq \sum_i r_i d(p_i, q_i)$$

2. $\mathbf{m}_{\underline{r}}$ *is* $PSL(2, \mathbb{R})-$*equivariant.*

3. $\mathbf{m}_{\underline{r}}$ *is diagonal, i.e.* $\mathbf{m}_{\underline{r}}(p, \cdots, p) = p$.

4. *If* σ *is a permutation of* I *then* $\mathbf{m}_{(r_{\sigma(1)}, \cdots, r_{\sigma(k)})}(p_{\sigma(1)}, \cdots, p_{\sigma(k)}) = \mathbf{m}_{\underline{r}}(p_1, \cdots, p_k)$.

Example III.17. Consider any two points $z, w \in \mathbb{H}^2$ and let $\underline{r} = (1/2, 1/2)$. Since barycenter map is equivariant we apply an isometry g that gz, gw lie on the imaginary axis with $gw = i$ and $\Im(z) > 1$. Then $\mathbf{m}_{(\underline{r})}(i, gz)$ is the point which minimizes the function

$$\Phi(p) = \frac{d(p, gz)^2 + d(p, i)^2}{2}.$$

Since $\frac{1}{2}d(p, gz)^2, \frac{1}{2}d(p, i)^2 \leq \Phi(p)$ it is easy to see that $\mathbf{m}_{\underline{r}}(i, gz)$ is just the midpoint of the geodesic $[i, gz]$. Using the equivariance it follows that $\mathbf{m}_{\underline{r}}(w, z)$ is the midpoint of the geodesic connecting $[w, z]$.

Proposition III.18 (Barycenter of Lipschitz Maps, Lemma 2.13 in [GK17])**.** *Let* I *and* $\underline{r}$ *be as in the previous proposition. Consider* $X \subset \mathbb{H}^2$, $p \in X$, *and Lipschitz maps* $f_i : X \to \mathbb{H}^2$ *for* $i \in I$. *The map* $f : X \to \mathbb{H}^2$ *defined by* $p \mapsto \mathbf{m}_{\underline{r}}(f_1(p), \cdots, f_k(p))$. *Then Lip*$(f) \leq \sum_i$ *Lip*(f_i).

Let $f_i : Y \to Y'$ be $K-$Lipschitz homeomorphisms between hyperbolic surfaces which are pairwise homotopic. Then let $\tilde{f}_i : \tilde{Y} \to \tilde{Y'}$ be their lifts such that $\tilde{f}_i(\tilde{y})$ lie in the same fundamental domain of Y'. Then if $\tilde{f} : \tilde{Y} \to \tilde{Y'}$ is the barycenter of $\tilde{f}_i$ with $\underline{r} = (1/k, \cdots, 1/k)$, then using the equivariance of $\tilde{f}_i$ and $\mathbf{m}$, this map can be pushed down to a Lipschitz map $f : Y \to Y'$ with Lipschitz constant less than K. This is well defined due to the equivariance of $\mathbf{m}$, as any other choice of lifts are related by a deck transform.

Let Y be a hyperbolic pair of pants and let μ be a maximal lamination on Y. Let $I_Y : Y \to Y$ be the map which cuts the pants into two isometric hexagons H_1 and H_2 and pastes them back together inside out. This isometry is orientation reversing and maps μ to another maximal lamination μ' of Y. See figure III.5. The shear parameters of the new maximal lamination μ' have the same magnitude as the shear parameters as μ but with the signs inverted as the sign is dependent on the orientation of the triangles about the geodesic changes.

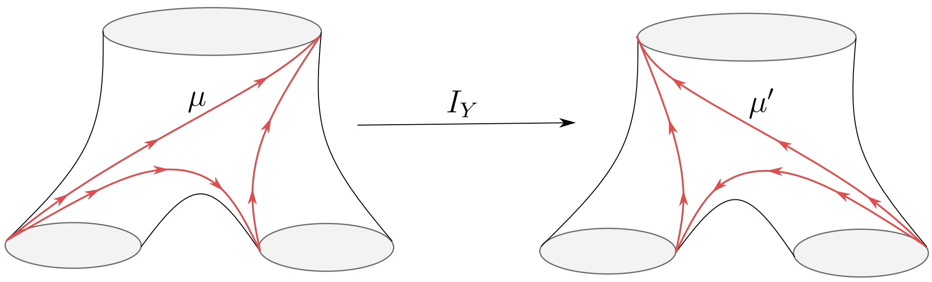

Figure III.5: The action of the map I_Y.

Definition III.19 (Construction of the average stretch map)**.** Let Y be a pair of pants and μ be some maximal lamination on Y. The we can construct another maximal lamination μ' on Y such that the shear parameters have the same magnitude and opposite sign. The there are stretch maps $\varphi_t^\mu : Y \to Y^\mu(t)$ and $\varphi_t^{\mu'} : Y \to Y^{\mu'}(t)$. Since $Y^{\mu'}$ and Y^μ are the same boundary lengths, the markings $(Y^\mu(t), \varphi_t^\mu f)$ and $(Y^{\mu'}(t), \varphi^{\mu'}(t) f)$ represent the same point in $\mathrm{Teich}(\Sigma)$, which will be denoted by just $\mathfrak{Y}(t)$. Define $\psi_t^\mu : Y \to Y^\mu(t)$ to be the barycenter of φ_t^μ and $I\varphi_t^{\mu'}$ with $\underline{r} = (1/2, 1/2)$, where $I : Y^{\mu'}(t) \to Y^\mu(t)$ is an isometry compatible with the markings.

Proposition III.20. *The map $\psi_t^\mu : Y \to Y(t)$ is an e^t-Lipschitz map.*

Proof. From proposition III.18 it follows that $\mathrm{Lip}(\psi_t^\mu) \leq e^t$. Since the stretch maps achieve the Lipschitz constant e^t at the boundary, it is easy to see that so does ψ_t^μ as all points in consideration lie along the same boundary geodesic:

$$\begin{aligned} d(\widetilde{\psi_t^\mu}(y_1), \widetilde{\psi_t^\mu}(y_2)) &= d(\widetilde{\psi_t^\mu}(y_1), \widetilde{\varphi_t^{\mu'}}(y_1)) \pm d(\widetilde{\varphi_t^{\mu'}}(y_1), \widetilde{\varphi_t^\mu}(y_2)) + d(\widetilde{\psi_t^\mu}(y_2), \widetilde{\varphi_t^\mu}(y_2)) \\ &= \frac{1}{2}\left(d(\widetilde{\varphi_t^\mu}(y_1), \widetilde{\varphi_t^\mu}(y_2)) + d(\widetilde{\varphi_t^{\mu'}}(y_1), \widetilde{\varphi_t^{\mu'}}(y_2)) \right) \\ &= e^t d(y_1, y_2). \end{aligned}$$

$\square$

Proposition III.21. *The map ψ_t^μ is an affine homeomorphism on the boundary of the pants.*

Proof. We prove a much more general statement which implies this claim. Suppose that iy_1, iy_2 are two points on the imaginary axis in $\mathbb{H}^2$. Then the barycenter of these two points weighed equally is the mid point along the geodesic joining them which in this case is $i\sqrt{y_1 y_2}$. An affine map on the imaginary axis is of the form $iy \mapsto iby^a$ where $a, b \in \mathbb{R}$. If we have any two affine maps on imaginary axis $iy \mapsto ib_1 y^{a_1}$ and $iy \mapsto ib_2 y^{a_2}$ then the barycenter map would be of the form $iy \mapsto i\sqrt{b_1 b_2} y^{\frac{a_1+a_2}{2}}$, which is again affine. This completes the proof as φ_t^μ and $\varphi_t^{\mu'}$ are affine maps on the boundary. $\square$

In order to prove that ψ_t^μ preserves the dual arc representatives we define a few tools and objects and re-state the problem in terms of these.

Definition III.22 (Teichmüller space of markings affine at boundary)**.** Let $[(Y_0, f_0 : \Sigma \to Y_0)] \in \text{Teich}(\Sigma)$. Denote the boundary components of Y_0 by $b_1, \cdots, b_n$. Then define

$$\begin{aligned}\text{Teich}_\partial(Y_0) = \{(Y, \mathbf{f} : Y_0 \to Y) \mid \mathbf{f} &\text{ is an orientation preserving homotopy equivalence,}\\ \mathbf{f}\big|_{b_i} &\text{ is an affine homeomorphism for each component}\}/\sim\end{aligned}$$

where $(Y, \mathbf{f}) \sim (W, \mathbf{g})$ if there is an isometry $I : Y \to W$ such that $I\mathbf{f}$ is homotopic to $\mathbf{g}$ relative to the boundary ∂Y_0, i.e. it fixes the boundary pointwise throughout the homotopy. We will denote elements in this space by $\mathfrak{Y}_\partial = [(Y, \mathbf{f})]_\partial$.

There is a natural map $\pi : \text{Teich}_\partial(Y_0) \to \text{Teich}(\Sigma)$ given by $[(Y, \mathbf{f})]_\partial \mapsto [(Y, \mathbf{f}f_0)]$.

Proposition III.23. *π is a surjective map.*

Proof. Given any $[(Y, f)] \in \text{Teich}(\Sigma)$ there is a "straight line" homotopy on the lift of boundary components in the universal cover from $\widetilde{ff_0^{-1}}\big|_{\tilde{b}_i}$ to an affine homeomorphism on $\tilde{b}_i$ for each i. This gives us a homotopy $H : \partial Y_0 \times [0,1] \to \partial Y$ between $ff_0^{-1}\big|_{\partial Y_0}$ and an affine homeomorphism on the boundary. By corollary 1.4 of [EK71] there is a homotopy $G : Y_0 \times [0,1] \to Y$ such that $G_0 = ff_0^{-1}$ and $H_t = G_t H_0$. The map $\mathbf{f} = G_1$ is affine on the boundary and is a homeomorphism. Then $\pi([(Y, \mathbf{f}_s)]_\partial) = \mathfrak{Y}$. $\square$

Suppose $\mathfrak{Y} = [(Y, f)] \in \text{Teich}(\Sigma)$ and without loss of generality assume that $ff_0^{-1} : Y_0 \to Y$ is affine on the boundary. Let $\underline{\alpha}(Y)$ and $\underline{\alpha}(Y_0)$ be the union of the special representatives of the dual arc system in Y and Y_0 respectively. Then $\Sigma - f_0^{-1}(\underline{\alpha}(Y_0))$ and $\Sigma - f^{-1}(\underline{\alpha}(Y))$ are both disjoint union of disks with punctures on the boundary and there are quotient maps q_0, q from these to Σ respectively. Cutting Σ along $f_0^{-1}(\underline{\alpha}(Y_0))$ and then applying the quotient map q on these collection of disks, then using the universal property of quotient maps there is a homeomorphism from $\Sigma \to \Sigma$ which maps $f_0^{-1}(\underline{\alpha}(Y_0))$ to $f^{-1}(\underline{\alpha}(Y))$. Composing with the markings this gives a homeomorphism $\mathbf{f}_s : Y_0 \to Y$ which preserves the representatives of the dual arcs. This defines a section $s : \text{Teich}(\Sigma) \to \text{Teich}_\partial(Y_0)$ which maps $[(Y, f)]$ to $[(Y, \mathbf{f}_s)]_\partial$.

Definition III.24 (Relative Twist parameter)**.** Suppose $(Y, \mathbf{f}), (W, \mathbf{g}) \in \pi^{-1}(\mathfrak{Y})$ and $I : Y \to W$ is an isometry such that $\varphi = I\mathbf{f}$ is homotopic to $\psi = \mathbf{g}$ via a homotopy H. Fix a lift $\tilde{\varphi}$ of φ. The homotopy H lifts to a unique homotopy in the universal cover $\tilde{H} : \tilde{Y}_0 \times I \to W$ so that $\tilde{H}_0 = \tilde{\psi}$. Consider the boundary component b_i of Y_0

and fix a lift of the boundary component $\tilde{b}_i$. For any $\tilde{y}_0 \in \tilde{Y}_0$ define the relative twist τ_i along b_i to be:

$$\tau_i((Y, \mathbf{f}), (W, \mathbf{g})) = d_s(\tilde{H}_0(\tilde{y}_0), \tilde{H}_1(\tilde{y}_0)) \tag{III.1}$$

where d_s is the signed distance along the orientation of the boundary components. Since H_1 is a lift of ψ we denote it by $\tilde{\psi}$.

Proposition III.25 (Relative Twist is well defined)**.** *The relative twist τ_i of the boundary b_i is independent of:*

1. *the point $\tilde{y}_0$.*
2. *the chosen lift of φ or the lift of the boundary component b_i.*
3. *the homotopy H.*
4. *the isometry I.*

Proof. 1. This follows directly from the fact that $\mathbf{f}$ and $\mathbf{g}$ are affine on the boundary. Since Y and W have the same boundary lengths it follows that $\mathbf{f}$ and $\mathbf{g}$ are just rotations. In the universal cover if we choose the lift of b_i to just be imaginary axis, then the affine maps are of the form $z \mapsto k_1 z$ and $z \mapsto k_2 z$ on the imaginary axis. Consider two points $\tilde{y}_0$ and $\tilde{y}_1$ in the lift $\tilde{b}_i$. Then,

$$d_s(\tilde{\varphi}(\tilde{y}_0), \tilde{\psi}(\tilde{y}_0)) = d_s(\tilde{\varphi}(\tilde{y}_0)), \tilde{\varphi}(\tilde{y}_1)) + d_s(\tilde{\varphi}(\tilde{y}_1)), \tilde{\psi}(\tilde{y}_1)) + d_s(\tilde{\psi}(\tilde{y}_1)), \tilde{\psi}(\tilde{y}_0))$$

from the discussion above the first and last terms cancel out.

2. A change in the lift just corresponds to composition with a deck transform which are isometries, and thus the choice of lift does not matter.

3. Consider two isotopies H and G between φ and ψ. Then we want to show that the unique lifts $\tilde{H}$ and $\tilde{G}$ with $\tilde{H}_0 = \tilde{G}_0 = \tilde{\varphi}$ for some fixed lift of φ, have the same endpoint i.e. $\tilde{H}_1 = \tilde{G}_1$. Consider the homotopy $F_t = (H_t)^{-1}(G_t)$ is a homotopy from $\mathrm{id}_{\tilde{Y}_0}$ to itself. For any y_0 in the boundary component b_i we get a closed loop $F_t(y_0)$. If this loop is non-trivial, then F_1 cannot be identity as any essential arc with both endpoints in b_i will not be mapped to itself by F_1. Thus $F_t(y_0)$ is a trivial loop on the boundary. By construction, we have $\tilde{H}_0(\tilde{y}_0) = \tilde{G}_0(\tilde{y}_0)$ and $\tilde{H}_1(\tilde{y}_0) = \gamma\tilde{G}_1(\tilde{y}_0)$ where γ is some deck transform. There is a unique lift $\tilde{F}$ so that $\tilde{F}_0 = \mathrm{id}_{\tilde{X}_0}$ which is given by $\tilde{F}_t = (\tilde{H}_t)^{-1}(\tilde{G}_t)$. Since $F_t(y_0)$ is a trivial loop it follows that $\tilde{F}_t(\tilde{y}_0)$ is a trivial loop in $\tilde{Y}_0$ and so $\tilde{F}_1 = \mathrm{id}_{\tilde{Y}_0}$ implying that $\tilde{H}_1(\tilde{y}_0) = \tilde{G}_1(\tilde{y}_0)$. This fixes the lift of φ.

4. This just follows from proposition I.26, as any other isometry compatible with the markings is homotopic to I and thus has the same lift to the universal cover. □

Proposition III.26 (Triangle identity)**.** *Let* $(Y_1, \mathbf{f}_1), (Y_2, \mathbf{f}_2), (Y_3, \mathbf{f}_3)$ *lie in the* $\pi-$*fiber of* $\mathfrak{Y} = [(Y, f)]$*. For any boundary component* b_i *of* Y_0 *the relative twist parameters satisfy the identity:*

$$\tau_i((Y_1, \mathbf{f}_1), (Y_3, \mathbf{f}_3)) = \tau_i((Y_1, \mathbf{f}_1), (Y_2, \mathbf{f}_2)) + \tau_i((Y_2, \mathbf{f}_2), (Y_3, \mathbf{f}_3)).$$

Proof. There are isometries $I_1 : Y_1 \to Y_2, I_2 : Y_2 \to Y_3$ which are compatible with the markings. If H is the homotopy from $I_1\mathbf{f}_1$ to $\mathbf{f}_2$ and G be a homotopy from $I_2\mathbf{f}_2$ to $\mathbf{f}_3$. Then the concatenation of the isotopies $F = G * (I_2 H)$ is a homotopy from $I_2 I_1 \mathbf{f}_1$ to $\mathbf{f}_3$. Using this homotopy to calculate $\tau_i((Y_1, \mathbf{f}_1), (Y_3, \mathbf{f}_3))$, the identity follows from the triangle inequality and the fact that all points lie on the same geodesic. □

It is easy to see that τ_i is a well defined map $\text{Teich}_\partial(Y) \times \text{Teich}_\partial(Y) \to \mathbb{R}$ because if $(Y, \mathbf{f}) \sim (W, \mathbf{g})$ in $\text{Teich}_\partial(Y)$ then their relative twist is zero as φ and ψ are equal on the boundary.

Proposition III.27. *Let* n *be the number of boundary components of* Y_0*. The map* $\underline{\tau}(\cdot, \mathfrak{Y}_\partial) : \pi^{-1}(\mathfrak{Y}) \to \mathbb{R}^n$ *which maps some* $\mathfrak{Y}'_\partial$ *to it's relative twist parameters is bijective.*

Proof. Suppose that $\tau_i((Y_1, \mathbf{f}_1), (Y, \mathbf{f})) = \tau_i((Y_2, \mathbf{f}_2), (Y, \mathbf{f}))$ for all boundary components b_i then using the triangle identity $\tau_i((Y_1, \mathbf{f}_1), (Y_2, \mathbf{f}_2)) = 0$. This means that the isometry $I : Y_1 \to Y_2$ is such that $I\mathbf{f}_1 = \mathbf{f}_2$ on the boundary. It is possible to construct a homotopy relative to the boundary between them as $\text{Homeo}_0(Y_0, \partial Y_0)$ is path connected. This proves injectivity of $\underline{\tau}$.
Let $(\tau_1, \cdots, \tau_n) \in \mathbb{R}^n$. Consider the lift $\tilde{\mathbf{f}}$ of $\mathbf{f} : Y_0 \to Y$ to the universal cover. On each boundary component $\mathbf{f}$ is affine, let $\widetilde{\partial \mathbf{f}_i}$ be the restriction on some lift $\tilde{b}_i$ of the boundary component b_i. There is a unique isometry $\gamma \in \text{PSL}(2, \mathbb{R})$ which fixes $\tilde{b}_i$ and $d_s(\mathbf{f}(\tilde{y}_0), \gamma\mathbf{f}(\tilde{y}_0)) = \tau_i$. Let $\widetilde{\partial \mathbf{g}_i}$ be $\gamma\widetilde{\partial \mathbf{f}_i}$ and let $\tilde{H}$ be a straight line homotopy between $\widetilde{\partial \mathbf{g}_i}$ and $\widetilde{\partial \mathbf{f}_i}$. Using the same technique as in III.23 we get a homeomorphism $\mathbf{g} : Y_0 \to Y$ such that $\mathbf{g}$ is homotopic to $\mathbf{f}$ and $\mathbf{g}\big|_{b_i} p_{Y_0} = p_Y \widetilde{\partial \mathbf{g}_i}$. It follows that $\tau_i((Y, \mathbf{g}), (Y, \mathbf{f})) = \tau_i$ for each b_i. This proves that $\underline{\tau}$ is surjective. □

Lemma III.28. *Let* $\mathfrak{Y} = [(Y, f)] \in$ *Teich*(Σ) *where* Σ *is the topological pair of pants. Let* μ, μ' *be a maximal laminations on* Y *as constructed above. Let* $\mathfrak{Y}^\mu_\partial(t) = [(Y^\mu(t), \varphi^\mu_t)]_\partial$, $\mathfrak{Y}^{\mu'}_\partial(t) = [(Y^{\mu'}(t), \varphi^{\mu'}_t)]_\partial$ *be points in Teich*$_\partial(Y)$ *where* φ^μ_t *and* $\varphi^{\mu'}_t$ *are* *the stretch maps $Y \to Y^{\mu}(t)$ and $Y \to Y^{\mu'}(t)$ with maximal laminations μ and μ' respectively. Let $\mathfrak{Y}_{\partial}(t) = [(Y^{\mu}(t), \psi_t^{\mu})]_{\partial}$ where ψ_t^{μ} is the average map as described above. Then the following are true:*

1. *For any boundary component b_i of Y,*
$$\tau_i(s(\mathfrak{Y}^{\mu}(t)), \mathfrak{Y}^{\mu}_{\partial}(t)) = -\tau_i(s(\mathfrak{Y}^{\mu}(t)), \mathfrak{Y}^{\mu'}_{\partial}(t)).$$

2. *For any boundary component b_i of Y,*
$$\tau_i(\mathfrak{Y}_{\partial}(t), s(\mathfrak{Y}^{\mu}(t))) = 0.$$

Proof. Since $\mathfrak{Y}^{\mu}(t) = \mathfrak{Y}^{\mu'}(t)$ as elements of $\text{Teich}(\Sigma)$ let $s(\mathfrak{Y}^{\mu}(t)) = s(\mathfrak{Y}^{\mu'}(t)) = [(Y^{\mu}(t), \mathbf{f}_s)]_{\partial}$. The idea for the proof of this lemma stems in the fact that φ_t^{μ} and $\varphi_t^{\mu'}$ are related to each other by an orientation reversing isometry, I_Y as described earlier. As I_Y maps μ to μ' it follows that the outer square in the following diagram commutes.

$$\begin{tikzcd}
Y \arrow[r, "I_Y"] \arrow[d, shift right, "\varphi_t^{\mu}"'] \arrow[d, shift left, "\mathbf{f}_s"] & Y \arrow[d, dashed, shift right, "\mathbf{g}_s"'] \arrow[d, shift left, "\varphi_t^{\mu'}"] \\
Y^{\mu}(t) \arrow[r, "I_{Y^{\mu}(t)}"'] & Y^{\mu'}(t)
\end{tikzcd}$$

Let $\mathbf{g}_s = I_{Y_{\mu}(t)}\mathbf{f}_s I_Y^{-1}$. Since I_Y pointwise preserves the seams along which we cut the pants into hexagons, and since $\mathbf{f}_s$ preserves the dual arc representatives, it follows that in each of the three possible cases for the dual arcs discussed in example III.2, $\mathbf{g}_s$ also preserves the dual arc representatives[1]. Using the commutativity of the outer square, and the fact that $\mathbf{f}_s \simeq \varphi_t^{\mu}$ it follows that $\mathbf{g}_s \simeq \varphi_t^{\mu'}$. It follows that $[(Y^{\mu'}(t), \mathbf{g}_s)]_{\partial} = [(Y^{\mu}(t), \mathbf{f}_s)]_{\partial}$ as the distance between $\tilde{\mathbf{g}}_s(\tilde{y})$ and $\tilde{\mathbf{f}}_s(\tilde{y})$ is zero where $\tilde{y}$ is the lift of a boundary point which belongs to the intersection of seam of the pants and some dual arc representative. With this construction the two statements follow.

1. For any boundary component b_i of Y let $\tilde{y}$ be the lift of a point y which is the endpoint of the seam of the pants on b_i. Then using the fact that I_Y is an orientation reversing isometry and $I_Y^{-1}(\tilde{y}) = \tilde{y}$:
$$\begin{aligned}
\tau_i((Y^{\mu'}(t), \mathbf{g}_s), (Y^{\mu'}(t), \varphi_t^{\mu'})) &= d_s(\widetilde{\mathbf{g}_s}(\tilde{y}), \widetilde{\varphi_t^{\mu'}}(\tilde{y})) \\
&= d_s(\widetilde{I_{Y^{\mu}(t)}\mathbf{f}_s I_Y^{-1}}(\tilde{y}), \widetilde{I_{Y^{\mu}(t)}\varphi_t^{\mu} I_Y^{-1}}(\tilde{y})) \\
&= -d_s(\widetilde{\mathbf{f}}(\tilde{y}), \widetilde{\varphi_t^{\mu}}(\tilde{y})) \\
&= -\tau_i((Y^{\mu}(t), \mathbf{f}_s), (Y^{\mu}(t), \varphi_t^{\mu}))
\end{aligned}$$

[1] The only non-trivial case to check is the last one. In this case the arc with endpoints on the same boundary is preserved by I_Y as it is just reflected about the midpoint.

Thus the first statement follows.

2. By definition of $\psi_t^\mu : Y \to Y^\mu(t)$ we know that for any $\tilde{y}$ in the lift of a boundary component b_i the signed distance $d_s(\widetilde{\psi_t^\mu}(\tilde{y}), \widetilde{\varphi_t^\mu}(\tilde{y})) = -d_s(\widetilde{\psi_t^\mu}(\tilde{y}), \widetilde{\varphi_t^{\mu'}}(\tilde{y}))$. Thus it follows that
$$\tau_i(\mathfrak{Y}_\partial(t), \mathfrak{Y}_\partial^\mu(t)) = -\tau_i(\mathfrak{Y}_\partial(t), \mathfrak{Y}_\partial^{\mu'}(t))$$
Thus using triangle identity and part 1 of the lemma, it follows that
$$\begin{aligned}\tau_i(\mathfrak{Y}_\partial(t), s(\mathfrak{Y}^\mu(t))) &= \tau_i(\mathfrak{Y}_\partial(t), \mathfrak{Y}_\partial^\mu(t)) + \tau_i(\mathfrak{Y}_\partial^\mu(t), s(\mathfrak{Y}^\mu(t))) \\ &= -(\tau_i(\mathfrak{Y}_\partial(t), \mathfrak{Y}_\partial^{\mu'}(t)) - \tau_i(\mathfrak{Y}_\partial^{\mu'}(t), s(\mathfrak{Y}^\mu(t)))) \\ &= -\tau_i(\mathfrak{Y}_\partial(t), s(\mathfrak{Y}^\mu(t))),\end{aligned}$$
which completes the proof.

□

This lemma tells us that ψ_t^μ preserves the dual arc representatives of $\mathfrak{Y}$ upto homotopy relative to the end points, and thus is a dilation map. Combining this with lemma III.14 this proves theorem III.15.

Remark. As mentioned in the above sections, dilation rays can be defined for arbitrary geodesic laminations as done by [CF21]. If λ is a geodesic lamination which is the Hausdorff limit of pants decomposition λ_n, then using results in [CF24] we can conclude that the dilation rays based at X with respect to λ are also Thurston geodesics.

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