$SL_2(\mathbb{R})/SO_2(\mathbb{R})$ IS HOMEOMORPHIC TO $\mathbb{H}$

MTH418: Presentation

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1 Defining the Map

Definition 1. We define the action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ as

$$A(z) = \frac{az+b}{cz+d}$$

where,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proposition 2. $SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$.

Proof. Let $z_0 = x_0 + iy_0 \in \mathbb{H}$. Then the map T_{z_0}

$$T_{z_0}(z) = y_0 z + x_0$$

maps i to z_0. Clearly T_{z_0} corresponds to the following matrix in $SL_2(\mathbb{R})$

$$\begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y_0} & 0 \\ 0 & \frac{1}{y_0} \end{pmatrix}.$$

Given any $z, w \in \mathbb{H}$ the matrix correponding to $T_z \circ T_w^{-1}$ maps w to z. □

Proposition 3. The stabilizer of i under this action is $SO_2(\mathbb{R})$.

Proof. Let $A \in SL_2(R)$ correspond to the map $A(z) = (az+b)/(cz+d)$. Then,

$$A(i) = i \iff i(a-d) = -(b+c) \iff a = d \;\&\; b = -c \iff A \in SO_2(\mathbb{R})$$

Thus $\mathrm{Stab}(i) = SO_2(\mathbb{R})$. □

Definition 4. Let $f : SL_2(\mathbb{R}) \to \mathbb{H}$ be defined as $f(A) = A(i)$.

Proposition 5. f is a continuous map.

Proof. Let A_n be a sequence $SL_2(\mathbb{R})$ which converges to A. If

$$f(A_n) = \frac{a_n i + b_n}{c_n i + d_n}$$

then $a_n \to a, b_n \to b, c_n \to c$, and $d_n \to d$ (using the induced metric on $SL_2(R)$ from $\mathbb{R}^4$), where $A(z) = (az+b)/(cz+d)$. Taking limits we get $\lim f(A_n) = f(A)$. □

Proposition 6. Let $\sim$ be an equivalence relation on $SL_2(\mathbb{R})$ given by $A \sim B \iff f(A) = f(B)$. Then $A\tilde{B}$ if and only if $B^{-1}A \in SO_2(\mathbb{R})$.

Proof. Let $A, B \in SL_2(\mathbb{R})$. Then,

$$A \sim B \iff A(i) = B(i) \iff B^{-1}A(i) = i \iff B^{-1}A \in SO_2(\mathbb{R}).$$

$\square$

The above theorem shows that $SL_2(\mathbb{R})/SO_2(\mathbb{R}) = SL_2(\mathbb{R})/\sim$. Thus as quotient space $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \simeq SL_2(\mathbb{R})/\sim$. From the previous proposition f is constant on the fibres of p. Thus we have the following diagram

$$\begin{tikzcd}
SL_2(\mathbb{R}) \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{H} \\
SL_2(\mathbb{R})/SO_2(\mathbb{R}) \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

where the map $\tilde{f}$ is induced by universal property of quotient maps, and is given by $\tilde{f}(A \cdot SO_2(\mathbb{R})) = A(i)$. Clearly this map is bijective: surjectivity follows from the transitive nature of the action, and injectivity follows from the previous proposition. The map $\tilde{f}$ is our candidate for the desired homeomorphism. If we can show that $\tilde{f}$ is also an open map we would be done.

2 Some results about Topological groups

Lemma 7. Let G be a locally compact, connected, Hausdorff topological group. Let V be an open neighborhood of $e \in G$. Then V generates G.

Proof. Let $H = \langle V \rangle$ be the subgroup generated by V. Then H is open since for any $g \in H$ the set $gV \subset H$ is open. Similarly given any $g \in H^c$ the open set $gV \subset H^c$ becasue if not, then there is some $v \in V$ such that $gv = h$ for some $h \in H$ hence $g = hv^{-1} \in H$, a contradiction! Since G is connected and H is a non-empty clopen subset of G it follows that $H = G$. $\square$

Proposition 8. Let G be as in Lemma 7. Let S be any subset of G such that $e \in S^\circ$. Then there exists a countable subset $E = \{g_i\}$ of G such that $\bigcup_i g_i S = G$.

Proof. Let $V \subset S^\circ$ such that $\overline{V}$ is compact and $e \in V$. For any $g \in G$, the element is contained in $W = V^{n_1} \cdots V^{n_k}$. The closure of W is contained in a compact set, and thus is compact. The collection $\{wV \mid w \in W\}$ is an open cover of W. By compactness, there is some finite subcover, say $\{w_i V \mid w_1, \cdots, w_n \in W\}$. Let $E' = w_i{}_{i=1}^{n}$. Since there are only countably many such W's and for each W there is a finite set E', the union of all such E' is a countable set. This union is our desired set E. $\square$

Note that the previous proposition works for any set S with non-empty interior since if $g \in S^\circ$ then $g^{-1}S$ contains e. Since $SL_2(\mathbb{R})$ is a connected Lie group it satisfies all conditions of Proposition 8. Thus given any set S with non-empty interior there exists a sequence A_i such that $\bigcup_i A_i S = SL_2(\mathbb{R})$. This can be projected to $SL_2(\mathbb{R})/SO_2(\mathbb{R})$. Given some set $\overline{S}$ with non-empty interior, $p^{-1}(\overline{S})$ also has a non-empty interior. Let the image of the countable set $\{A_i\}$ be $\{\bar{A}_i\}$. Then

$$
\begin{aligned}
\bigcup_{i=1}^{\infty} p(A_i p^{-1}(\bar{S})) &= p(SL_2(\mathbb{R})) = SL_2(\mathbb{R})/SO_2(\mathbb{R}) \\
\bigcup_{i=1}^{\infty} p(A_i) p(p^{-1}(\bar{S})) &= SL_2(\mathbb{R})/SO_2(\mathbb{R}) \\
\bigcup_{i=1}^{\infty} \bar{A}_i \bar{S} &= SL_2(\mathbb{R})/SO_2(\mathbb{R})
\end{aligned}
$$

We formulate this as the following proposition.

Proposition 9. Let S be any subset of $\bar{G} = SL_2(\mathbb{R})/SO_2(\mathbb{R})$ such that $S^\circ \neq \emptyset$. Then there exists a countable subset $E = \{\bar{A}_i\}$ of $\bar{G}$ such that $\bigcup_i \bar{A}_i S = \bar{G}$.

3 $\tilde{f}$ is an open map

Before proving the final result, I'll state the Baire Category theorem for metric spaces.

Theorem 10 (Baire Category Theorem)**.** Let X be a metric space. Then any countable union of closed sets of X with empty interior also has empty interior.

Proposition 11. $\tilde{f}$ is an open map.

Proof. Let $\bar{G}$ be $SL_2(\mathbb{R})/SO_2(\mathbb{R})$.
Step 1. Fix $\bar{A} \in \bar{G}$. Let K be a compact neighborhood of $\bar{A}$ with non-empty interior, then $\tilde{f}(K)^\circ \neq \emptyset$. Suppose not, then there is some compact neighborhood K with $K^\circ \neq \emptyset$ but $\tilde{f}(K)^\circ = \emptyset$. By Proposition 9 there exists $\bar{A}_i$ such that $\bigcup_i \bar{A}_i K^\circ = \bar{G}$. Applying $\tilde{f}$ to this,

$$\bigcup_i \tilde{f}(\bar{A}_i K^\circ) = \mathbb{H}$$

Since $\tilde{f}(\bar{A}_i K^\circ)$ has empty interior, this contradicts Baire Category theorem (since $\mathbb{H}$ is a metric space).
Step 2. If $T \in SL_2(\mathbb{R})$ and $\bar{A} \in \bar{G}$ then $\tilde{f}(T\bar{A}) = T\tilde{f}(\bar{A})$ and similarly for $\tilde{f}^{-1}$. This also implies that $\tilde{f}(T \cdot S) = T\tilde{f}(S)$ for any subset S.
Step 3. Let $\mathcal{B} = \{\tilde{f}^{-1}(\tilde{f}(K)^\circ) \mid \forall \text{ compact } K \text{ s.t. } K^\circ \neq \emptyset\}$. We claim that these are a basis for the topology of $\bar{G}$. Let $\bar{A} \in \bar{G}$. Then there exists some compact neighborhood K of $\bar{A}$ such that K° is non-empty (by local compactness). Thus $U = \tilde{f}^{-1}(\tilde{f}(K)^\circ) \subset K^\circ$ is an open set. In case $\bar{A} \notin U$ then we can simply translate it by left actions so that

$\bar{A} \in T \cdot U = \tilde{f}^{-1}(\tilde{f}(T \cdot K)^{\circ})$ for some $T \in SL_2(\mathbb{R})$. Suppose that U and U' in $\mathcal{B}$ are neighborhoods of $\bar{A}$ then

$$U \cap U' = \tilde{f}^{-1}(\tilde{f}(K)^{\circ} \cap \tilde{f}(K')^{\circ}) = \tilde{f}^{-1}(\tilde{f}(K \cap K')^{\circ}) \in \mathcal{B}.$$

Hence $\mathcal{B}$ is a basis of *some* topology. To show that it generates the quotient topology we will prove that inside every neighborhood V of $\bar{A}$ there is a element of $\mathcal{B}$. Since $\bar{G}$ is locally compact, it follows that there is some open neighborhood W of $\bar{A}$ such that $\bar{W} \subset V$ is compact. It follows that:

$$\begin{aligned} \bar{W} &\subset V \\ \tilde{f}(\bar{W})^{\circ} &\subset F(V) \\ \tilde{f}^{-1}(\tilde{f}(\bar{W})^{\circ}) &\subset V. \end{aligned}$$

Due to continuity we know that $U = \tilde{f}^{-1}(\tilde{f}(\bar{W})^{\circ}) \subset W^{\circ} \subset \bar{W}$. In case $\bar{A} \notin U$ then we can translate it by some T so that $\bar{A} \in T \cdot U$. If $T \cdot U$ does not completely lie inside V then we just consider $W' = \bigcup_T T^{-1}(T \cdot U \cap W) \subset U$, where the union is over all T such that TU is a neighborhood of $\bar{A}$. Repeating the previous calculation we get an open set $U' = \tilde{f}^{-1}(\tilde{f}(\overline{W'})^{\circ}) \subset W'$ which when translated by T such that $\bar{A} \in T \cdot U$ is a subset of V. Since $\bar{A} \in T \cdot U \in \mathcal{B}$ it follows that $\mathcal{B}$ is a basis for the quotient topology.

Step 4. Since $\tilde{f}(U)$ for all $U \in \mathcal{B}$ is open by construction, it follows that $\tilde{f}$ is an open map. $\square$